arXiv:math/0601400v3 [math.KT] 7 Mar 2006

THE STEINBERG GROUP OF A MONOID RING, NILPOTENCE, AND ALGORITHMS

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ABSTRACT. For a regular ring R and an affine monoid M the homotheties of M act nilpotently on the Milnor unstable groups of $R[M]$. This strengthens the K_2 part of the main result of [G5] in two ways: the coefficient field of characteristic 0 is extended to any regular ring and the stable K_2-group is substituted by the unstable ones. The proof is based on a polyhedral/combinatorial technique, computations in Steinberg groups, and a substantially corrected version of an old result on elementary matrices by Mushkudiani [Mu]. A similar stronger nilpotence result for K_1 and algorithmic consequences for factorization of high Frobenius powers of invertible matrices are also derived.

1. Introduction

1.1. **Main result.** In the recent work [G5] we proved the following result. Let $\mathbf{k}$ be a field of characteristic 0, M be an additive submonoid of $\mathbb{Z}^n$ without nontrivial units, and i be a nonnegative integer. Then for any element $x \in K_i(\mathbf{k}[M])$ and any natural number $c \geq 2$ there exists an integer $j_x \geq 0$ such that $(c^j)_*(x) \in K_i(\mathbf{k})$ for all $j \geq j_x$.

Here for a natural number c the group endomorphism of $K_i(\mathbf{k}[M])$, induced by the monoid endomorphism $M \to M$, $m \mapsto m^c$, is denoted by c_*.

The motivation for this result is that it is a natural higher version of the triviality of algebraic vector bundles on affine toric varieties [G1], contains Quillen's fundamental result on homotopy invariance, and easily extends to global toric varieties. See the introduction of [G5] for the details.

This result confirms the *nilpotence conjecture* for a special class of coefficients rings. The conjecture asserts the similar nilpotence property of higher K-groups of monoid algebras over *any (commutative) regular* coefficient ring.

The main result in this paper is a stronger unstable version of the nilpotence property for the functors $K_{1,r}$ and $K_{2,r}$ for any regular coefficient ring. Moreover, when the coefficient ring is a field the argument leads to an algorithm for factorization of high 'Frobenius powers' of invertible matrices into elementary ones.

In the special case of the polynomial rings $\mathbf{k}[\mathbb{Z}_+^n] = \mathbf{k}[t_1, \dots, t_n]$ the algorithmic study of factorizations of invertible matrices has applications in signal processing [LiXW, PW]. The starting point here is Suslin's well known paper [Su]. In this special case there is no need to take Frobenius powers of invertible matrices. However, a K-theoretical obstruction shows that this is no longer possible once we leave the

2000 *Mathematics Subject Classification.* 14M25, 19B14, 19C09, 20G35, 52B20.

class of free monoids, see Remark 2.5. Therefore, our algorithmic factorization is an optimal 'sparse version' of the existing algorithm for polynomial rings.

Here is the main result:

Theorem 1.1. *Let M be a commutative cancellative torsion free monoid without nontrivial units, $c \geq 2$ a natural number, R a commutative regular ring and $\mathbf{k}$ a field. Then:*

(a) *For any element $z \in K_{2,r}(R[M])$, $r \geq \max(5, \dim R + 3)$, there exists an integer $j_z \geq 0$ such that*
$$(c^j)_*(z) \in K_{2,r}(R) = K_2(R), \qquad j \geq j_z.$$

(b) *For any matrix $A \in \mathrm{GL}_r(R[M])$, $r \geq \max(3, \dim R + 2)$, there exists an integer number $j_A \geq 0$ such that*
$$(c^j)_*(A) \in \mathrm{E}_r(R[M])\,\mathrm{GL}_r(R), \qquad j \geq j_A.$$

(c) *There is an algorithm which for any matrix $A = \mathrm{SL}_r(\mathbf{k}[M])$, $r \geq 3$, finds an integer number $j_A \geq 0$ and a factorization of the form:*
$$(c^{j_A})_*(A) = \prod_k e_{p_k q_k}(\lambda_k), \qquad e_{p_k q_k}(\lambda_k) \in \mathrm{E}_r(\mathbf{k}[M]).$$

Here:

- for a commutative ring Λ its Krull dimension is denoted by $\dim \Lambda$,
- $K_{2,r}(-)$ refers to the Milnor's rth unstable K_2,
- for a natural number c the group endomorphisms $\mathrm{GL}_r(R[M]) \to \mathrm{GL}_r(R[M])$ and $K_{2,r}(R[M]) \to K_{2,r}(R[M])$, induced by the monoid endomorphism $M \to M$, $m \mapsto m^c$, are both denoted by c_*.
- for two subgroup H_1 and H_2 of a group G we use the notation $H_1H_2 = \{h_1h_2 \mid h_1 \in H_1,\ h_2 \in H_2\}$.

Remark 1.2. We do not give a detailed description of the actual algorithm, mentioned in Theorem 1.1(c). Rather, throughout the text, we highlight the explicit nature of the proof of Theorem 1.1(b) which implies the possibility of converting the argument into an implemented algorithm when the coefficients are in a field.

Remark 1.3. It is not difficult to show that the proof of Theorem 1.1, given below, works for a more general class of rings of coefficients. In fact, all one needs from the ring R is the validity of the claims (a), (b) and (c) for the polynomial extension $R[t_1, \dots, t_n]$, $n = \operatorname{rank} M$ – a classically known fact when R is a regular ring, see Section 2.

Remark 1.4. We do not know whether there is a uniform bound j_i, depending on M, R and i, but not on $x \in K_i(R[M])$, such that $(c^{j_i})_*(x) \in K_i(R)$. Nontrivial examples in [G3] indicate that such bounds may in fact exist, at least for K_1.

A word is in order on the previous results and the proof of Theorem 1.1.

The proof of the nilpotence of $K_i(\mathbf{k}[M])$ as given in [G5] – even in the case of Milnor's K_2 – uses a series of deep facts in higher K-theory of rings, obtained from

the early 1990s on (the most recent of which is [Cor]). The proof of Theorem 1.1, given below, makes no use of any of these results. It is based on computations in $\mathrm{E}_r(R[M])$, essentially due to Mushkudiani [Mu], and similar computations in $\mathrm{St}_r(R[M])$. The explicit nature of these computations is also the source of the algorithmic consequences for $\mathrm{SL}_r(\mathbf{k}[M])$. Obviously, no such a pure algebraic approach is possible for higher K-groups.

Actually, the weaker stable version of Theorem 1.1(a) for K_2 is claimed in [Mu] and the present work grew up from our attempts to understand Mushkudiani's argument. Eventually, what survived from [Mu] is his preliminary computations in the group of elementary matrices – an important technical fact whose corrected and stronger unstable version is given in the last Section 8; see Remarks 5.4, 6.3 and 6.5.[1] The rest of the paper is devoted to the reduction of Theorem 1.1 to this technical fact.

In the course of the proof we also develop an effective/algorithmic excision technique for the unstable K_1 and K_2-groups of monoid rings (Section 4). It allows us to circumvent Suslin-Wodzicki's excision theorem [SuW] – a result which is applicable only to stable groups and which was essential in [G5].

Finally, a comment on the result on K_1: the weaker stable analog of Theorem 1.1(b) is obtained in [G2], where we originally conjectured the nilpotence of the higher K-theory of $R[M]$. But the essential difference between the two approaches is that in the present paper we never invoke *Quillen's local-global patching*, *Karoubi squares* and *Horrock's localizations at monic polynomials*, heavily used in [G1, G2, G5]. On the other hand, it should be mentioned that the technique developed in [G2] is crucial in the proof of the nilpotence result for higher K-groups, see [G4, §9].

1.2. **Organization of the paper.** To make the exposition as self-contained as possible, the necessary K-theoretical background, together with a further motivation for the main result, is provided in Section 2. In Section 3 we give a quick summary of the polyhedral approach to commutative, cancellative, torsion free monoids, developed in our study of K-theory of monoid rings. An effective excision technique for unstable K_1- and K_2-groups of monoid rings is developed in Section 4. In Section 5 we introduce an inductive process, *pyramidal descent*, on which the proof of Theorem 1.1 is based. The main technical facts that make this inductive process work, Theorems 6.1 and 6.4, are stated in Section 6. There we also explain how 6.4 follows from 6.1. In Section 7 we show the validity of pyramidal descent in the situation of Theorem 1.1. Section 8 presents a corrected version of Mushkudiani's proof of Theorem 6.1.

Acknowledgment. I am grateful to the referee for the thorough study of the paper and spotting various inaccuracies in the original version, especially in Section 8.

[1] We also greatly simplify the notation in [Mu] – already a challenge on its own right.

2. K-theoretical background

Let Λ be a ring and $r \geq 2$ a natural number. For a pair of natural numbers $1 \leq p, q \leq r$ and an element $\lambda \in \Lambda$ the matrix with λ on the pq-position and 0s elsewhere will be denoted $a_{pq}(\lambda)$.

The standard elementary matrices over Λ of order r are defined as follows

$$e_{pq}(\lambda) = \mathbf{1} + a_{pq}(\lambda), \quad 1 \leq p, q \leq r, \quad p \neq q, \quad \lambda \in \Lambda,$$

where $\mathbf{1}$ is the unit matrix.

The standard elementary matrices generate *the subgroup of elementary matrices* $\mathrm{E}_r(\Lambda)$ inside the general linear group $\mathrm{GL}_r(\Lambda)$ of order r.

Starting from now on all our rings are assumed to be *commutative*.

It is known that $\mathrm{E}_r(\Lambda) \subset \mathrm{GL}_r(\Lambda)$ is a normal subgroup as soon as $r \geq 3$ [Su].

The *special linear group* $\mathrm{SL}_r(\Lambda)$ *of order* r is defined to be the subgroup of $\mathrm{GL}_r(\Lambda)$ of the matrices with determinant 1. Thus $\mathrm{E}_r(\Lambda) \subset \mathrm{SL}_r(\Lambda) \subset \mathrm{GL}_r(\Lambda)$.

Let G_r denote any of the groups $\mathrm{E}_r(\Lambda)$, $\mathrm{SL}_r(\Lambda)$, $\mathrm{GL}_r(\Lambda)$. *The stable group* G is defined to be the inductive limit of the diagram of groups

$$\begin{aligned} &G_2(\Lambda) \to \cdots \to G_r(\Lambda) \to G_{r+1}(\Lambda) \to \cdots, \\ &A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \qquad A \in G_r(\Lambda). \end{aligned}$$

The *Whitehead Lemma* says that $\mathrm{E}(\Lambda) = [\mathrm{GL}(\Lambda), \mathrm{GL}(\Lambda)]$ [Mi, Lemma 3.1]. *The Bass-Whitehead group* $K_1(\Lambda)$ is defined by

$$K_1(\Lambda) = \mathrm{GL}(\Lambda)/\,\mathrm{E}(\Lambda) = \mathrm{GL}(\Lambda)_{\mathrm{ab}} = H_1(\mathrm{GL}(\Lambda), \mathbb{Z}).$$

Its unstable versions are given by $K_{1,r}(\Lambda) = \mathrm{GL}_r(\Lambda)/\,\mathrm{E}_r(\Lambda)$, $r \geq 3$.

The standard elementary matrices satisfy *the Steinberg relations*:

$$\begin{aligned} e_{pq}(\lambda) \cdot e_{pq}(\mu) &= e_{pq}(\lambda + \mu), \\ [e_{pq}(\lambda), e_{qu}(\mu)] &= e_{pu}(\lambda\mu), \quad p \neq u \\ [e_{pq}(\lambda), e_{uv}(\mu)] &= 1, \quad p \neq v, \ q \neq u. \end{aligned}$$

The unstable Steinberg group $\mathrm{St}_r(\Lambda)$ (over Λ) is defined by the generators $x_{pq}(\lambda)$, $1 \leq p, q \leq r$, $p \neq q$ and $\lambda \in \Lambda$, subject to the corresponding Steinberg relations. The stable group $\mathrm{St}(\Lambda)$ is the inductive limit of the diagram $\mathrm{St}_2(\Lambda) \to \mathrm{St}_3(\Lambda) \to \cdots$.

The Milnor r-th unstable group $K_{2,r}(\Lambda)$ is defined as the kernel of the canonical surjective group homomorphism $\mathrm{St}_r(\Lambda) \to \mathrm{E}_r(\Lambda)$. Passing to the inductive limits we get the short exact sequence of the corresponding stable groups:

$$1 \to K_2(\Lambda) \to \mathrm{St}(\Lambda) \to \mathrm{E}(\Lambda) \to 1.$$

This is the sequence of a universal central extension of the perfect group $\mathrm{E}(\Lambda)$ [Mi, Theorem 5.10]. Consequently, $K_2(\Lambda) = H_2(\mathrm{E}(R), \mathbb{Z})$.

Van der Kallen has shown [K2] that the extension

$$1 \to K_{2,r}(\Lambda) \to \mathrm{St}_r(\Lambda) \to \mathrm{E}_r(\Lambda) \to 1.$$

is also universal central if $r \geq 5$.

All groups mentioned above, stable or unstable, depend functorially on the underlying ring Λ.

Theorem 2.1. *Let R be a regular ring. Then $K_i(R) = K_i(R[t_1, \dots, t_n])$, $i = 1, 2$, for all natural numbers n.*

Theorem 2.1 is true for all indices $i = 0, 1, 2, \dots$[2] The case $i = 0$ is due to Grothendieck, the case $i = 1$ is due to Bass-Heller-Swan [BaHS], and the general case $i \geq 2$ is due to Quillen [Q1].

Theorem 2.2 ([Su])**.** *Let R be a noetherian ring with $\dim R < \infty$ and n be a nonnegative integer. Then the natural homomorphisms*

$$K_{1,r}(R[t_1, \dots, t_n]) \to K_1(R[t_1, \dots, t_n])$$

are surjective for $r \geq \max(2, \dim R + 1)$ and bijective for $r \geq \max(3, \dim R + 2)$.

Theorems 2.1 and 2.2 have the following

Corollary 2.3. *Let $\mathbf{k}$ be a field and n be a natural number. Then*

$$\mathrm{SL}_r(\mathbf{k}[t_1, \dots, t_n]) = \mathrm{E}_r(\mathbf{k}[t_1, \dots, t_n]), \qquad r \geq 3.$$

Suslin proves this equality in [Su] directly, without invoking the Bass-Heller-Swan isomorphism. This is done by developing a K_1-analog of *Quillen's local-global patching* and *Horrocks' monic inversion technique*, the two crucial ingredients in Quillen's proof of Serre's conjecture on projective modules [Q2]. It is exactly Suslin's proof of Corollary 2.3 what is used in the algorithm, developed in [PW]:

Theorem 2.4 ([PW])**.** *Let $\mathbf{k}$ be a field and n be a natural number. There is an algorithm which for any matrix $A \in \mathrm{SL}_r(\mathbf{k}[t_1, \dots, t_n])$ finds a factorization of the form:*

$$A = \prod_k e_{p_k q_k}(\lambda_k), \qquad \lambda_k \in \mathbf{k}[t_1, \dots, t_n].$$

Remark 2.5. The inequality $r \geq 3$ is sharp as shown by the following example of Cohn [Coh]. For any field $\mathbf{k}$ we have

$$A = \begin{pmatrix} 1 + t_1 t_2 & -t_1^2 \\ t_2^2 & 1 - t_1 t_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbf{k}[t_1, t_2]) \setminus \mathrm{E}_2(\mathbf{k}[t_1, t_2]).$$

By Corollary 2.3, A becomes an elementary matrix already in $\mathrm{SL}_3(\mathbf{k}[t_1, t_2])$. However, if we consider the monomial ring $\mathbf{k}[t_1^2, t_1 t_2, t_2^2]$ over which A is defined, then the matrix A represents a non-zero element in $K_1(\mathbf{k}[t_1^2, t_1 t_2, t_2^2])$, [G3, Example 8.2]. Therefore, A does not become an elementary matrix in any of the groups $\mathrm{SL}_r(\mathbf{k}[t_1^2, t_1 t_2, t_2^2])$, no matter how large r is. This explains the relevance of Frobenius actions (that is, the homomorphisms c_*) in the nilpotence conjecture.

[2] and for noncommutative regular rings as well.

Remark 2.6. For a field $\mathbf{k}$ one can sandwich the 2-dimensional polynomial rings between two copies of $\mathbf{k}[t_1^2, t_1t_2, t_2^2]$ as follows

$$\mathbf{k}[t_1^2, t_1t_2, t_2^2] \subset \mathbf{k}[t_1, t_2] \subset \mathbf{k}[t_1, t_1^{1/2}t_2^{1/2}, t_2] \cong \mathbf{k}[t_1^2, t_1t_2, t_2^2].$$

This observation and Corollary 2.3 show that $(2_*)(A) \in \mathrm{E}_r(\mathbf{k}[t_1^2, t_1t_2, t_2^2])$ for all $r \geq 3$. An elaborated version of this argument, in combination with an excision technique, implies Theorem 1.1(b,c) in the special case when M is a *simplicial monoid,* which means $M \subset \mathbb{Z}^n$ is a finitely generated additive submonoid and the cone in $\mathbb{R}^n$ spanned by M is simplicial; see Corollaries 4.3 and 4.4 below. However, the existence of such a sandwiched polynomial ring

$$\mathbf{k}[M] \subset \mathbf{k}[\mathbb{Z}_+^n] \subset \mathbf{k}[M^{1/c}], \qquad M^{1/c} = \{m^{1/c} \mid m \in M\} \subset \mathbb{Z}\left[\frac{1}{c}\right] \otimes \mathrm{gp}(M)$$

implies that M is simplicial. This partly explains why the general case of the nilpotence conjecture is essentially more difficult than the simplicial case.

Tulenbaev's result below, proved in [T], is a K_2-analog of Suslin's work [Su].

Theorem 2.7. *Let R be a noetherian ring of finite Krull dimension* $\dim R$ *and n a natural number. Then the natural homomorphisms*

$$K_{2,r}(R[t_1, \dots, t_n]) \to K_2(R[t_1, \dots, t_n])$$

are surjective for $r \geq \max(4, \dim R + 2)$ *and bijective for* $r \geq \max(5, \dim R + 3)$.

Earlier van der Kallen had shown that $K_{2,r}(R) = K_2(R)$ for $r \geq \dim R + 3$ [K1]. Correspondingly, we will always write $K_2(R)$ instead of $K_{2,r}(R)$ when r is as in Theorem 2.7.

3. Monoids and cones

Here is a quick summary of the generalities on cones and monoids. For more detailed account the interested reader is referred to [BrG, Chapters 1, 2].

3.1. **Polytopes and cones.** A *polytope* $P \subset \mathbb{R}^n$ means the convex hull of finitely many points in $\mathbb{R}^n$. This is the same as a compact intersection of finitely many affine half-spaces in $\mathbb{R}^n$. For a polytope $P \subset \mathbb{R}^n$ its relative interior will be denoted by $\mathrm{int}(P)$. A polytope $P \subset \mathbb{R}^n$ is called *rational* if it is spanned by rational points. A polytope P is rational if and only if it is a compact intersection of finitely many affine half-spaces whose boundaries are rational affine hyperplanes. A polytope is a *simplex* if it is the convex hull of an *affinely independent system* of points.

The set of nonnegative reals is denoted by $\mathbb{R}_+$. For a subset $X \subset \mathbb{R}^n$ we will use the notation $\mathbb{R}_+X = \{\sum_i a_ix_i \mid a_i \in \mathbb{R}_+,\ x_i \in X\}$.

A *cone* $C \subset \mathbb{R}^n$ means a subset of the form $\mathbb{R}_+X \subset \mathbb{R}^n$ where X is finite. This is the same as the intersection of a finite family of halfspaces in $\mathbb{R}^n$ whose boundary hyperplanes are linear subspaces of $\mathbb{R}^n$. When $X \subset \mathbb{Q}^n$ (equivalently, the mentioned halfspaces have rational boundary hyperplanes) the cone is called *rational.* A cone is *pointed* if it contains no pair of opposite nonzero vectors. A cone $C \subset \mathbb{R}^n$ can be embedded (via a linear map) in $\mathbb{R}^{\dim C}$. If C is rational then such an embedding can

be chosen to be rational. Further, a cone $C \subset \mathbb{R}^n$ is pointed if and only if it can be embedded in the positive orthant $\mathbb{R}_+^{\dim C}$.

All our cones will be assumed to be pointed.

Let $C \subset \mathbb{R}^n$ be a cone and $\mathcal{H}^+ \subset \mathbb{R}^n$ be a half-space, defined by an inequality $\xi_1 X_1 + \cdots + \xi_n X_n \geq 0$, such that $C \subset \mathcal{H}^+$. Let $\mathcal{H}$ be the boundary hyperplane $\xi_1 X_1 + \cdots + \xi_n X_n = 0$. Then the intersection $C \cap \mathcal{H}$ is called a *face of* C. The origin 0 and the cone C itself are the smallest and the biggest faces of C. A *facet* of a cone $C \subset \mathbb{R}^n$ is a maximal proper face, which is the same as a codimension 1 face. The boundary ∂C is defined as the union of all proper faces of C, and *the relative interior* $\operatorname{int}(C)$ is defined by $\operatorname{int}(C) = C \setminus \partial C$.

A *d-cone* means a d-dimensional cone.

An *open cone* in $\mathbb{R}^n$ of dimension d is by definition the union of the relative interiors of d-cones, forming a nested system of cones, plus the origin 0.

An *affine cone* means a parallel translate of a cone.

For a rational d-cone $C \subset \mathbb{R}^n$, $d > 0$, there always exists a rational affine $(n-1)$-dimensional subspace $\mathcal{G} \subset \mathbb{R}^n \setminus \{0\}$ such that $C = \mathbb{R}_+(C \cap \mathcal{G})$ or, equivalently, $C \cap \mathcal{G}$ is a rational $(d-1)$-polytope. For such a pair C and $\mathcal{G}$ we write $\Phi(C) = C \cap \mathcal{G}$. Further, for a real number $\varepsilon > 0$ we will use the notation $C(\varepsilon) = \mathbb{R}_+\Phi(C)(\varepsilon)$ where $\Phi(C)(\varepsilon)$ is the ε-neighborhood of $\Phi(C)$ in $\mathcal{G}$. Thus $C(\varepsilon) \subset \mathbb{R}^n$ is an n-dimensional open cone.

A cone is called *simplicial* if it is spanned by a system linearly independent vectors, or equivalently, the polytope $\Phi(C)$ is a simplex.

3.2. **Monoids.** A *monoid* will always mean a commutative, cancellative, torsion free monoid. Equivalently, our monoids are additive submonoids of rational vector spaces.

Our blanket assumption on the notation of monoid operation is that when a monoid is considered inside its monoid ring we use multiplicative notation. Otherwise we use additive notation.

For a monoid M its group of differences will be denoted by $\operatorname{gp}(M)$. We put $\operatorname{rank} M = \operatorname{rank} \operatorname{gp}(M)$. If a monoid is finitely generated then it is called *affine*. Thus an affine monoid is, up to isomorphism, a finitely generated additive submonoid of $\mathbb{Z}^n$. Moreover, whenever appropriate we can without loss of generality assume that $\operatorname{gp}(M) = \mathbb{Z}^n$.

A monoid is called *positive* if its group of invertible elements is trivial. For an affine positive monoid $M \subset \mathbb{Z}^n$ the subset $\mathbb{R}_+M \subset \mathbb{R}^n$ is a rational cone. A monoid M is called *simplicial* if it is positive, affine and the cone $\mathbb{R}_+M$ is simplicial.

For an affine positive monoid $M \subset \mathbb{Z}^n$, $\operatorname{rank} M > 0$, and an affine hyperplane $\mathcal{G} \subset \mathbb{R}^n$ such that $\mathbb{R}_+M = \mathbb{R}_+(\mathbb{R}_+M \cap \mathcal{G})$, we will use the notation $\Phi(M)$ for $\mathbb{R}_+M \cap \mathcal{G}$. For a convex subset $W \subset \Phi(M)$ we introduce the submonoid

$$M|W = M \cap \mathbb{R}_+W \subset M.$$

If W consists of a single point p then we write $M|p$ instead of $M|\{p\}$.

For M and $\mathcal{G}$ as above we will also use the notation $M_* = M \cap \mathbb{R}_+\text{int}(\Phi(M))$[3] and $M|F = M \cap F \subset M$ for $F \subset \mathbb{R}_+M$ a face. Thus $M_* = \big(M \cap \text{int}(\mathbb{R}_+M)\big) \cup \{0\}$. More generally, if $N \subset M$ is any (not necessarily affine) submonoid then we put $\Phi(N) = \mathcal{G} \cap \mathbb{R}_+N$ and

$$N_* = N \cap \mathbb{R}_+\{x \mid x \text{ is in the relative interior of } \Phi(N)\}.$$

For an affine positive monoid $M \subset \mathbb{Z}^n$ and a convex subset $W \subset \Phi(M)$ (w.r.t. to an appropriately fixed hyperplane $\mathcal{G} \subset \mathbb{R}^n$ as above) it is easily shown that

$$\dim W = \operatorname{rank} M - 1 \implies \operatorname{gp}(M) = \operatorname{gp}(M|W). \tag{1}$$

(See, for instance, [BrG, Corollary 2.25].) In particular,

$$\operatorname{gp}(M) = \operatorname{gp}(M_*). \tag{2}$$

Let $M \subset \mathbb{Z}^n$ be an affine positive monoid, $F \subset \mathbb{R}_+M$ a face, and R a ring. Then we have the R-algebra retraction:

$$\pi_F : R[M] \to R[M|F], \quad \pi(m) = \begin{cases} m \text{ if } m \in M|F, \\ 0 \text{ if } m \in M \setminus (M|F). \end{cases}$$

A monoid M is called *normal* if $kx \in M$ implies $x \in M$ for any $x \in \operatorname{gp}(M)$ and any $k \in \mathbb{N}$. Any affine positive normal monoid of rank n is up to isomorphism of the form $C \cap \mathbb{Z}^n$ where $C \subset \mathbb{R}^n$ is a positive rational n-cone. Conversely, any such an intersection $C \cap \mathbb{Z}^n$ is always an affine positive normal monoid. The finite generation part of the latter claim is classically known as *Gordan's lemma* ([BrG, Lemma 2.7]).

For any monoid M there is the smallest submonoid of $\operatorname{gp}(M)$ – the *normalization of M* – which is normal and contains M:

$$\bar{M} = \{x \in \operatorname{gp}(M) \mid kx \in M \text{ for some natural number } k\}.$$

For an affine normal positive monoid $M \subset \mathbb{Z}^n$ and a convex subset $W \subset \mathcal{G}$, where $\mathcal{G} \subset \mathbb{R}^n$ is a hyperplane cross-secting $\mathbb{R}_+M$, we introduce the monoid:

$$M|W = \operatorname{gp}(M) \cap \mathbb{R}_+W.$$

When $W \subset \Phi(M)$ this notation is compatible with the one introduced above for not necessarily normal monoids.

A monoid M is called *seminormal* if the following implications holds:

$$x \in \operatorname{gp}(M),\ 2x \in M,\ 3x \in M \implies x \in M.$$

Lemma 3.1. *Let $M \subset \mathbb{Z}^n$ be an affine positive monoid. Then M is seminormal if and only if the monoid $(\bar{M|F})_*$ is normal for any face $F \subset \mathbb{R}_+M$. Moreover, if M is seminormal then $M_* = \bar{M}_*$.*

The first part is proved in [G1] (for not necessarily affine monoids), see also [BrG, Proposition 2.37]. The second part follows from the equality (2).

[3] Here we follow the convention that the interior of a point is the point itself. In particular, $M = M_*$ when $\operatorname{rank} M = 1$.

3.3. **Divisible monoids.** For a natural number c and a monoid M we say that M is c-*divisible* if for any element $z \in M$ the equation $cx = z$ is solvable for x inside M. Since our monoids are cancellative and torsion free, such a solution is unique.

For a monoid M and a natural number c the submonoid of $\mathbb{Z}\left[\frac{1}{c}\right] \otimes \operatorname{gp}(M)$, generated by $\frac{1}{c} \otimes x$, $x \in M$, will be denoted by M/c.

For a natural number $c \geq 2$ the c-*divisible hull* of M is defined as the filtered union

$$M/c^{\infty} = \bigcup_{j=1}^{\infty} M/c^{j} \subset \mathbb{Q} \otimes \operatorname{gp}(M).$$

It is easily checked that for a natural number $c \geq 2$ all c divisible monoids L are seminormal:

$$2x, 3x \in L \implies cx \in L \implies x = \frac{1}{c} \cdot (cx) \in L.$$

By Lemma 3.1 the submonoid $M_*/c^{\infty} \subset M/c^{\infty}$ is a normal monoid for any positive affine monoid M. It easily follows that for any affine positive monoid M we have:

$$(M_*)^{c^{-\infty}} = (\bar{M}_*)^{c^{-\infty}}. \tag{3}$$

When M is simplicial much more is true:

Proposition 3.2. *Let M be an affine simplicial monoid. Then for any finite subset $S \subset M_*/c^{\infty}$ one can effectively find a free submonoid $L \subset M_*/c^{\infty}$ such that $S \subset L$. In particular, M_*/c^{∞} is a filtered union of free monoids.*

Without effective nature of the claim this is Theorem A in [G2]. However, what is proved in [G2] is literally what is stated above.

Next we derive a structural result on c-divisible monoids that will be used in Section 8.3. Let $M \subset \mathbb{Z}^n$ be an affine positive monoid and let $h : \mathbb{Z}^n \to \mathbb{Z}$ be a surjective group homomorphism. Then M carries the *graded structure:*

$$M = \cdots \cup M_{-1} \cup M_0 \cup M_1 \cup \cdots, \qquad M_i = M \cap h^{-1}(i).$$

('Graded' here means $M_i + M_j \subset M_{i+j}$ and $M_i \cap M_j = \emptyset$ whenever $i \neq j$.) For an element $m \in M_i$ we will write $\deg(m) = i$.

For simplicity of notation we let the same h denote the $\mathbb{R}$-linear extension $\mathbb{R} \otimes h : \mathbb{R}^n \to \mathbb{R}$.

Lemma 3.3. *Let $M \subset \mathbb{Z}^n$ be an affine positive monoid with* $\operatorname{gp}(M) = \mathbb{Z}^n$. *Let $m \in M_*$ with* $\deg(m) = d \neq 0$. *Then one can effectively find a decomposition of the form:*

$$m = \sum_{i=1}^{|d|} m_i, \quad m_i \in \begin{cases} (M_*)^{c^{-\infty}} \cap h^{-1}(1) & \text{if } d > 0, \\ (M_*)^{c^{-\infty}} \cap h^{-1}(-1) & \text{if } d < 0. \end{cases}$$

Proof. We consider the case $d > 0$ and the other case is symmetric.

Consider the broken line $\mathfrak{a} = [0a_1a_2 \dots a_{d-1}m]$ in $\mathbb{R}^n$, obtained by subdividing the segment $[0, m] \subset \mathbb{R}^n$ into d equal parts. This broken line can be though of as the decomposition inside $\mathbb{Q}^n$:

$$m = d^{-1}m + \cdots + d^{-1}m.$$

We want to find (effectively!) a broken line in $\mathbb{R}^n$

$$\mathfrak{m} = [m_0m_1m_2 \dots m_d], \quad m_0 = 0, \quad m_d = m,$$

satisfying the condition $m_i - m_{i-1} \in (M_*)^{c^{-\infty}} \cap h^{-1}(1)$ for $i = 1, \dots, d-1$. Since $\mathbb{R}_+M_*$ is an open cone, any broken line $\mathfrak{b} = [m_0m_1m_2 \dots m_{d-1}m_d]$ that is obtained from $\mathfrak{a}$ by an arbitrary sufficiently small perturbation of the vertices $a_1, \dots, a_{d-1}$ will satisfy the condition $m_i - m_{i-1} \in \mathbb{R}_+M_*$ for $i = 1, \dots, d-1$. Therefore, it is enough to show that for every index $i \in \{1, \dots, d-1\}$ the affine real hyperplane $h^{-1}(i) \subset \mathbb{R}^n$ contains elements of $(M_*)^{c^{-\infty}}$ arbitrarily close to a_i. In view of the equalities (1) and (3), it is enough to show that for every index $i \in \{1, \dots, d-1\}$ the affine real hyperplane $h^{-1}(i) \subset \mathbb{R}^n$ contains elements of $\mathrm{gp}(M)^{c^{-\infty}}$ arbitrarily close to a_i. This will be done by showing that for every $i \in \{1, \dots, d-1\}$ the set $\mathrm{gp}(M)^{c^{-\infty}} \cap h^{-1}(i)$ is dense in the affine hyperplane $h^{-1}(i) \subset \mathbb{R}^n$.

The conditions $\mathrm{gp}(M) = \mathbb{Z}^n$ and $h(\mathbb{Z}^n) = \mathbb{Z}$ imply that the sets

$$h^{-1}(i) \cap \mathrm{gp}(M), \quad i = 1, \dots, d-1,$$

are cosets of $\mathrm{Ker}(h) \cap \mathbb{Z}^n$ in $\mathbb{Z}^n$. In particular,

$$\mathrm{gp}(M)^{c^{-\infty}} \cap h^{-1}(i) \simeq (\mathbb{Z}[1/c])^n \cap \mathrm{Ker}(h), \quad i = 1, \dots, k-1,$$

where $\mathbb{Z}[1/c]$ refers to the localization of the ring of integers $\mathbb{Z}$ at c and $\simeq$ refers to the isometry equivalence w.r.t. the Euclidean metric. But $(\mathbb{Z}[1/c])^n \cap \mathrm{Ker}(h)$ is a c-divisible rank$(n-1)$ subgroup of $\mathrm{Ker}(h) \cong \mathbb{R}^{n-1}$. In particular, it is a dense subset of $\mathrm{Ker}(h)$.

The algorithmic aspect of Lemma 3.3 follows from the fact that we can effectively compute (in terms of generators) the group $\mathbb{Z}^n \cap \mathrm{Ker}(h)$, its appropriate cosets in $\mathbb{Z}^n$, and find an element of $\mathrm{gp}(M)^{c^{-\infty}} \cap \mathrm{Ker}(h)$ in any explicitly given neighborhood in $\mathrm{Ker}(h)$. □

The multiplicative counterpart of the notation M/c^j and M/c^∞, to be used in monoid rings, is $M^{c^{-j}}$ and $M^{c^{-\infty}}$.

The relevance of c-divisible monoids is explained by the following equivalent reformulation of Theorem 1.1:

Theorem 3.4. *Let M, c, R and $\mathbf{k}$ be as in Theorem 1.1. Then*

(a) $K_2(R) = K_{2,r}(R[M^{c^{-\infty}}])$ *for* $r \geq \max(5, \dim R + 3)$.
(b) $\mathrm{GL}_r(R[M^{c^{-\infty}}]) = \mathrm{E}_r(R[M^{c^{-\infty}}])\,\mathrm{GL}_r(R)$ *for* $r \geq \max(3, \dim R + 2)$.

(c) *There is an algorithm which for any matrix* $A = \mathrm{SL}_r(\mathbf{k}[M])$*,* $r \geq 3$*, finds an integer number* $j_A \geq 0$ *and a factorization of the form:*

$$A = \prod_k e_{p_k q_k}(\lambda_k), \qquad e_{p_k q_k}(\lambda_k) \in \mathrm{E}_r(\mathbf{k}[M^{c^{-j_A}}]).$$

In the subsequent sections we will freely use the equivalence between the two formulations.

Remark 3.5. Essentially, c-divisible monoids enter our argument through Proposition 3.2 (and a variation of it – Lemma 4.5) and Lemma 3.3, used correspondingly in Sections 4 and 8. They also partially explain why in this paper we mainly work with open cones. In [G5] the importance of c-divisible monoids is related to the excision results in [SuW] and that of open cones – to Karoubi squares of certain type.

4. Reduction to interior monoids

Proposition 4.1. *Let* $M \subset \mathbb{Z}^n$ *be an affine positive monoid. Assume Theorem 1.1 is valid for the submonoids of the form* $(M|F)_* \subset M$ *where* $F \subset \mathbb{R}_+ M$ *is a facet or* $F = \mathbb{R}_+ M$*. Then the theorem is valid also for* M*.*

For a matrix $A \in \mathrm{GL}_r(R[M])$ the elements $m \in M$ that show up in the canonical R-linear expansion of its entries will be called the *support monomials of* A.

A monoid is a filtered union of its affine submonoids. Moreover, one can find effectively such a filtered union representation for any explicitly given monoid. Therefore, by the equality (3) in Section 3.3, Proposition 4.1 and the equivalent reformulation of Theorem 1.1 in Theorem 3.4 we get

Corollary 4.2. *For Theorem 1.1 it is enough to show that for any affine positive normal monoid* M *we correspondingly have:*

(a) $K_2(R) = K_{2,r}(R[(M_*)^{c^{-\infty}}])$*,*
(b) $\mathrm{GL}_r(R[(M_*)^{c^{-\infty}}]) = \mathrm{E}_r(R[(M_*)^{c^{-\infty}}])\,\mathrm{GL}_r(R)$*,*
(c) *There is an algorithm that for any matrix* $A \in \mathrm{SL}_r(\mathbf{k}[M_*])$ *finds an integer number* $j_A \geq 0$ *and a factorization of the form:*

$$A = \prod_k e_{p_k q_k}(\lambda_k), \qquad e_{p_k q_k}(\lambda_k) \in \mathrm{E}_r(\mathbf{k}[(M_*)^{c^{-j_A}}]).$$

(Here r *is as in the corresponding part of Theorem 1.1.)*

In the next three subsections we prove Proposition 4.1, considering the three parts of Theorem 1.1 separately and in the reversed order. The case of Milnor groups requires substantially more work.

4.1. **The case of Theorem 1.1(c).** Let $F \subset \mathbb{R}_+ M$ be a facet and $A \in \mathrm{SL}_r(\mathbf{k}[M])$. Consider the matrix $A|F = \pi_F(A) \in \mathrm{SL}_r(\mathbf{k}[M|F])$. Obviously, $A|F$ is effectively computable from A: its support monomials are those of A that belong to F. By the assumption, $A|F$ can be effectively factored into standard elementary matrices over $\mathbf{k}[(M|F)^{c^{-j_F}}]$ for some explicitly computable $j_F \in \mathbb{N}$. Therefore, it is enough

to prove Theorem 1.1(c) for the matrix $A_F = (A|F)^{-1}A \in \mathrm{SL}_r(\mathbf{k}[M])$. Observe that no support monomial of A_F belongs to $M|F$.

Now let $G \subset \mathbb{R}_+M$ be another facet. Again by the assumption the matrix $A_F|G = \pi_G(A_F) \in \mathrm{SL}_r(\mathbf{k}[M|G])$ can be algorithmically factored into elementary matrices over the ring $\mathbf{k}[(M|G)^{c^{-j_G}}]$ for some explicitly computable $j_G \in \mathbb{N}$. It is enough to prove Theorem 1.1(c) for the matrix $A_{F,G} = (A_F|G)^{-1}A_G \in \mathrm{SL}_r(\mathbf{k}[M])$.

The crucial observation at this point is that no support monomial of the matrix $A_{F,G}$ belongs to $(M|F) \cup (M|G)$.

Continuing the process until all facets of the cone $\mathbb{R}_+M$ are considered, we arrive at a matrix

$$A_{F,G,\dots,H} \in \mathrm{SL}_r(\mathbf{k}[M_*])$$

where $\{F, G, \dots, H\}$ is the set of facets of $\mathbb{R}_+M$. By the assumptions in the proposition, one can find $j_M \in \mathbb{N}$ and a factorization of $A_{F,G,\dots,H}$ into standard elementary matrices from $\mathrm{E}_r(\mathbf{k}[(M_*)^{c^{-j_M}}])$.

It is then clear that the desired explicit factorization of A can be found over the ring $\mathbf{k}[M_*^{c^{-j_F-j_G-\dots-j_H-j_M}}]$. □

In view of Theorem 2.4 and Proposition 3.2 the induction on $\operatorname{rank} M$ yields

Corollary 4.3. *Theorem 1.1(c) is true for simplicial monoids.*

4.2. **The case of Theorem 1.1(b).** Essentially the same argument as above goes through. In more detail, consider a matrix $A \in \mathrm{GL}_r(R[M])$. For a facet $F \subset \mathbb{R}_+M$ we have the matrix $A|F = \pi_F(A) \in \mathrm{GL}_r(R[M|F])$. By the assumptions in the proposition, there exists $E_F \in \mathrm{E}_r(R[(M|F)^{c^{-\infty}}]) \subset \mathrm{E}_r(R[M^{c^{-\infty}}])$ such that $E_F \cdot (A|F) \in \mathrm{GL}_r(R)$. In particular, $E_F \in \mathrm{GL}_r(R[M|F])$ and no support monomial of the matrix E_FA belongs to $M|F$.

It is enough to show that $E_FA \in \mathrm{E}_r(R[M^{c^{-\infty}}])\,\mathrm{GL}_r(R)$.

Consider another facet $G \subset \mathbb{R}_+M$. Again by the induction hypothesis there exists $E_G \in \mathrm{E}_r(R[(M|G)^{c^{-\infty}}]) \subset \mathrm{E}_r(R[M^{c^{-\infty}}])$ such that $E_G \cdot ((E_FA)|G) \in \mathrm{GL}_r(R)$. In this situation $E_G \in \mathrm{GL}_r(R[M|G])$ and no support monomial of E_GE_FA belongs to $M|G$. We claim that no support monomial of E_GE_FA belongs to $M|F$ too. In fact, we have $\pi_F(E_GE_FA) = \pi_F(E_G)\pi_F(E_FA) \subset \mathrm{GL}_r(R[M|G])\,\mathrm{GL}_r(R)$. In particular, if there were a support monomial of E_GE_FA in $M|F$ then it would also belong to $M|G$. But such does not exist.

Continuing the process with the remaining facets we find a system of elementary matrices

$$E_F \in \mathrm{E}_r(R[(M|F)^{c^{-\infty}}]),\ E_G \in \mathrm{E}_r(R[(M|G)^{c^{-\infty}}]), \dots, E_H \in \mathrm{E}_r(R[(M|H)^{c^{-\infty}}]),$$
$$F, G, \dots, H \subset \mathbb{R}_+M \text{ are the facets,}$$

such that $E_H \cdots E_GE_FA \in \mathrm{GL}_r(R[M_*])$. But over $R[M_*]$ we are done by the assumptions in the proposition. □

In view of Theorems 2.1, 2.2 and Proposition 3.2 the induction on $\operatorname{rank} M$ yields

Corollary 4.4. *Theorem 1.1(b) is true for simplicial monoids.*

4.3. **The case of Theorem 1.1(a).** This is not as straightforward as the previous cases.

Lemma 4.5. *Let $c \geq 2$ be a natural number and M_1, M_2 be c-divisible monoids of rank 1 without nontrivial units. Then the submonoid*

$$N = M_1 \times M_2 \setminus \{(a, 0) \mid a \in M_1,\ a \neq 0\} \subset M_1 \times M_2$$

is a filtered union of rank 2 free monoids.

Proof. There are inductive systems of indices I and J and elements $a_i \in M_1$, $i \in I$, and $b_j \in M_2$, $j \in J$, such that:

- $M_1 = \bigcup_I A_i$ and $M_2 = \bigcup_J B_j$,
- $A_i = \mathbb{Z}_+ a_i$ and if $i_1 < i_2$ then $s_{i_2,i_1} a_{i_2} = a_{i_1}$ for some natural number $s_{i_2,i_1} \geq 2$,
- $B_j = \mathbb{Z}_+ b_j$ and if $j_1 < j_2$ then $t_{j_2,j_1} b_{j_2} = b_{j_1}$ for some natural number $t_{j_2,j_1} \geq 2$.

For any pair $(i, j) \in I \times J$ consider the monoid

$$N_{ij} = \mathbb{Z}_+(a_i, b_j) + \mathbb{Z}_+(0, b_j) \cong \mathbb{Z}_+^2$$

For any indices $i \in I$ and $j_1, j_2 \in J$ with $j_1 \leq j_2$ we have

$$(a_i, b_{j_1}) = (a_i, b_{j_2}) + (t_{j_2,j_1} - 1)(1, b_{j_2}) \in N_{ij_2}.$$

Therefore, $N_{ij_1} \subset N_{ij_2}$. In particular, the monoids

$$N_i = A_i \times M_2 \setminus \{(a, 0) \mid a \in A_i,\ a \neq 0\}$$

are filtered unions of the monoids N_{ij}, $j \in J$. But N is a filtered union of the monoids N_i. $\square$

In the next lemma we use the following notation: for a homomorphism of rings $\Lambda_1 \to \Lambda_2$ and a natural number r we let $\mathrm{St}_r^*(\Lambda_1)$ denote the image of the map $\mathrm{St}_r(\Lambda_1) \to \mathrm{St}_r(\Lambda_2)$.

Lemma 4.6. *Let R be a regular ring of finite Krull dimension d and $r \geq \max(5, d+3)$. Assume c, M_1, M_2 and $N \subset M_1 \times M_2$ are as in Lemma 4.5. Then arbitrary element $w \in \mathrm{St}_r(R[M_1 \times M_2])$ admits a presentation of the form*

$$w = uv, \quad u \in \mathrm{St}_r^*(R[M_1]), \quad v \in \mathrm{St}_r^*(R[N]).$$

(Here the maps from $\mathrm{St}_r(R[M_1])$ and $\mathrm{St}_r(R[N])$ to $\mathrm{St}_r(R[M_1 \times M_2])$ are the ones induced by the identity ring embeddings $R[M_1] \to R[M_1 \times M_2]$ and $R[N] \to R[M_1 \times M_2]$.)

Proof. Consider the commutative square of R-algebra homomorphisms whose horizontal arrows are identity embeddings:

$$\begin{tikzcd} R[N] \arrow[r] \arrow[d, "\vartheta|_{R[N]}"'] & R[M_1 \times M_2] \arrow[d, "\vartheta"] \\ R \arrow[r] & R[M_1] \end{tikzcd}, \quad \vartheta|_{M_1} = \mathbf{1}_{M_1}, \quad \vartheta(M_2 \setminus \{1\}) = 0.$$

Because $\mathrm{St}_r(R) \to \mathrm{St}_r(R[M_1]) \to \mathrm{St}_r(R[M_1 \times M_2])$ are (split) injective homomorphisms, we can identify $\mathrm{St}_r(R)$ and $\mathrm{St}_r(R[M_1])$ with the subgroups $\mathrm{St}_r^*(R) \subset \mathrm{St}_r^*(R[M_1]) \subset \mathrm{St}_r(R[M_1 \times M_2])$.

Let $\tau : \mathrm{St}_r^*(R[N]) \to \mathrm{St}_r(R)$ be the homomorphism induced by the augmentation $R[M_1 \times M_2] \to R$, $M_1 \times M_2 \setminus \{(1,1)\} \to 0$.

First we show the following inclusion

$$\mathrm{Ker}(\tau)\,\mathrm{St}_r(R[M_1]) \subset \mathrm{St}_r(R[M_1])\mathrm{Ker}(\tau). \tag{4}$$

Assume $u_1 \in \mathrm{St}_r(R[M_1])$ and $v_1 \in \mathrm{Ker}(\tau)$. We want to prove that $v_1 u_1 \in \mathrm{St}_r(R[M_1])\mathrm{Ker}(\tau)$.

Let $v' = u_1^{-1} v_1 u_1 \in \mathrm{St}_r(R[M_1 \times M_2])$ and e_1 and e' be the images of v_1 and v' in $\mathrm{E}_r(R[M_1 \times M_2])$. From the commutative square

$$\begin{tikzcd}
\mathrm{St}_r^*(R[N]) \arrow[r, "\tau"] \arrow[d] & \mathrm{St}_r(R) \arrow[d] \\
\mathrm{E}_r(R[M_1 \times M_2]) \arrow[r, "\mathrm{E}_r(\vartheta)"] & \mathrm{E}_r(R[M_1])
\end{tikzcd}$$

we see that $e_1 \in \mathrm{Ker}(\mathrm{E}_r(\vartheta))$. Then $e' \in \mathrm{Ker}(\mathrm{E}_r(\vartheta))$ as well. In particular, $e' \in \mathrm{SL}_r(R[N])$.

By Lemma 4.5 N is a filtered union of rank 2 monoids. Therefore, by Theorems 2.1 and 2.2 we have $\mathrm{GL}_r(R[N]) = \mathrm{GL}_r(R)\,\mathrm{E}_r(R[N])$ and so

$$e' \in \mathrm{Ker}(\mathrm{E}_r(\vartheta)) \cap \mathrm{GL}_r(R)\,\mathrm{E}_r(R[N]) \subset \mathrm{E}_r(R[N]).$$

Let $v'' \in \mathrm{St}_r^*(R[N])$ be a preimage of e'. There exists $z \in K_{2,r}(R[M_1 \times M_2])$ such that $v' = zv''$. The monoid $M_1 \times M_2$ is clearly a filtered union of rank 2 free monoids and so $K_{2,r}(R[M_1 \times M_2]) = K_2(R)$ by Theorems 2.1 and 2.7. Hence the desired representation

$$v_1 u_1 = u_2 v_2, \quad u_2 = u_1 z \tau(v'') \in \mathrm{St}_r(R[M_1]), \quad v_2 = \tau(v'')^{-1} v'' \in \mathrm{Ker}(\tau).$$

Finally, Lemma 4.6 follows from (4) because any generator $x_{ij}(\lambda)$ of the group $\mathrm{St}_r(R[M_1 \times M_2])$ has a representation of the form:

$$x_{ij}(\lambda) = x_{ij}(\vartheta(\lambda)) x_{ij}(\lambda - \vartheta(\lambda)),\ x_{ij}(\vartheta(\lambda)) \in \mathrm{St}_r(R[M_1]),\ x_{ij}(\lambda - \vartheta(\lambda)) \in \mathrm{Ker}(\tau).$$

$\square$

From now on we assume that c, R, r and M are as in Theorem 1.1(a).

Fix a facet $F \subset \Phi(M)$. By the induction hypothesis we have

$$K_{2,r}(R[(M|F)^{c^{-\infty}}]) = K_2(R). \tag{5}$$

Any element $z \in K_{2,r}(R[M])$ has a representation of the form $z = \prod_k v_k$ where:

- $v_{k_1} \in \mathrm{St}_r^*(R[(M|p_{k_1})^{c^{-\infty}}]), \ldots, v_{k_s} \in \mathrm{St}_r^*(R[(M|p_{k_s})^{c^{-\infty}}])$,
- $v_k \in \mathrm{St}_r^*(R[(M|q_k)^{c^{-\infty}}])$, $k \notin \{k_1, \ldots, k_s\}$,
- $k_1 < \ldots < k_s$, $\quad p_{k_1}, \ldots, p_{k_s} \in F$, $q_k \in \Phi(M) \setminus F$, $k \notin \{k_1, \ldots, k_s\}$.

(For instance, any representation of the form $\prod_k x_{i_k j_k}(\mu_k)$ where $\mu_k \in RM^{c^{-\infty}}$ is of this form.)

When $\{k_1, \ldots, k_s\} \neq \emptyset$ we say that z has a representation of $(k_1, \ldots, k_s)$-*type*.

Lemma 4.7. *If $z \in K_{2,r}(R[M^{c^{-\infty}}])$ has a representation of $(1, \ldots, s)$-type then*

$$z \in \operatorname{Im}(K_{2,r}(R[(M|\Phi(M) \setminus F)^{c^{-\infty}}]) \to K_{2,r}(R[M^{c^{-\infty}}])).$$

Proof. Let $z = \prod_k v_k$ be a representation of $(1, \ldots, s)$-type. Then, denoting by $e_k \in \mathrm{E}_r(R[M^{c^{-\infty}}])$ the image of v_k, $k = 1, \ldots, s$, we have

$$\prod_{k \leq s} e_k = \left(\prod_{k>s} e_k\right)^{-1} \in \mathrm{E}_r(R[(M|F)^{c^{-\infty}}]) \cap \mathrm{E}_r(R[(M|\Phi(M) \setminus F)^{c^{-\infty}}]) \subset$$
$$\mathrm{SL}_r(R) \cap \mathrm{E}_r(R[(M|F)^{c^{-\infty}}]) = \mathrm{E}_r(R)$$

(The latter equality follows from the fact that R is a retract of $R[(M|F)^{c^{-\infty}}]$.) In particular, there exists an element $z_1 \in \mathrm{St}_r(R)$ such that $zz_1^{-1} \in K_{2,r}(R[(M|F)^{c^{-\infty}}]) = K_2(R)$ (by (5)). Now the lemma follows because

$$z = \left(zz_1^{-1}\right)\left(z_1 \prod_{k>s} v_k\right) \in \operatorname{Im}(K_{2,r}(R[(M|\Phi(M) \setminus F)^{c^{-\infty}}]) \to K_{2,r}(R[M^{c^{-\infty}}])).$$

□

Lemma 4.8. *If $z \in K_{2,r}(R[M^{c^{-\infty}}])$ has a representation of $(k_1, \ldots, k_s)$-type for some $(k_1, \ldots, k_s) \neq (1, \ldots, s)$ then z has a representation of $(l_1, \ldots, l_s)$-type for some $(l_1, \ldots, l_s) < (k_1, \ldots, k_s)$ w.r.t. the lexicographical order.*

Proof. Let $z = \prod_k v_k$ be a representation of $(k_1, \ldots, k_s)$-type and $i \in \{1, \ldots, s\}$ be the smallest index with $i < k_i$. Thus $(k_1, \ldots, k_s) = (1, 2, \ldots, i-1, k_i, k_{i+1}, \ldots, k_s)$. (We do not exclude the case when $i = 1$.)

In this situation we have $v_{k_i - 1} \in \mathrm{St}_r^*(R[(M|q)^{c^{-\infty}}])$ for some $q \in \Phi(M) \setminus F$ and $v_{k_i} \in \mathrm{St}_r^*(R[(M|p)^{c^{-\infty}}])$ for some $p \in F$. By Lemma 4.6 we can write

$$z = \left(\prod_{k<k_i-1} v_k\right) \cdot (v_{k_i-1} v_{k_i}) \cdot \left(\prod_{k \geq k_i+1} v_k\right) = \left(\prod_{k<k_i-1} v_k\right) \cdot (uv) \cdot \left(\prod_{k \geq k_i+1} v_k\right)$$

for some $u \in \mathrm{St}_r^*(R[(M|p)^{c^{-\infty}}])$ and $v \in \mathrm{St}_r^*(R[\big(M|[q,p)\big)^{c^{-\infty}}])$. Here $[q, p)$ refers to the corresponding half-open segment in $\Phi(M)$.

There exists a representation of the form $v = \prod_j w_j$ where $w_j \in \mathrm{St}_r^*(R[(M|t_j)^{c^{-\infty}}])$ for some $t_j \in \Phi(M) \setminus F$. Then

$$z = \left(\prod_{k<k_i-1} v_k\right) \cdot u \cdot \left(\prod_j w_j\right) \cdot \left(\prod_{k \geq k_i+1} v_k\right)$$

is a representation of $(1, 2, \ldots, i-1, k_i - 1, k'_{i+1}, \ldots, k'_s)$-type for some $k'_{i+1} \geq k_{i+1}$, $\ldots$, $k'_s \geq k_s$. □

By Lemmas 4.7 and 4.8 we have

Corollary 4.9. *The identity embedding* $R[(M|(\Phi(M)\setminus F))^{c^{-\infty}}] \to R[M^{c^{-\infty}}]$ *induces a surjective homomorphism*

$$\iota_F : K_{2,r}(R[(M|(\Phi(M)\setminus F))^{c^{-\infty}}]) \to K_{2,r}(R[M^{c^{-\infty}}]).$$

Now we complete the proof of Proposition 4.1 as follows.

Consider a facet $F \neq G \subset \Phi(M)$. Applying the same argument as in the proof of Corollary 4.9 to the elements of $\mathrm{Im}(\iota_F)$ we arrive to the conclusion that the natural homomorphism

$$\iota_{F,G} : K_{2,r}(R[(M|(\Phi(M)\setminus (F\cup G)))^{c^{-\infty}}]) \to K_{2,r}(R[(M|(\Phi(M)\setminus F))^{c^{-\infty}}])$$

is also surjective. Then we consider another facet of $\Phi(M)$ etc. Finally we obtain the surjectivity of the composite homomorphism

$$\iota_{F,G,\dots,H} : K_{2,r}(R[(M_*)^{c^{-\infty}}]) \to K_{2,r}(R[M^{c^{-\infty}}]).$$

But $K_{2,r}(R[(M_*)^{c^{-\infty}}]) = K_2(R)$. $\square$

In view of Theorems 2.1, 2.7 and Proposition 3.2 the induction on $\operatorname{rank} M$ yields

Corollary 4.10. *Theorems 1.1(a) is true for simplicial monoids.*

5. Pyramidal descent

In this section we introduce a polyhedral induction technique in K-theory of monoid rings, called *pyramidal descent*, here adapted to the situation of Theorem 1.1. It was introduced in [G1] and further refined in [G5]. We in fact need the refinement of the technique as developed in [G5], see Remark 5.4.

5.1. **Pyramidal extensions of polytopes.** A polytope $P \subset \mathbb{R}^n$ is called a *pyramid* if it is a convex hull of one of its facets $F \subset P$ and a vertex $v \in P$, not in the affine hull of F. In this situation F is a *base* and v is an *apex* of P, and we write $P = \mathrm{pyr}(v, F)$. For instance, an arbitrary simplex is a pyramid such that every facet is a base and every vertex is an apex.

The *complexity* of a d-dimensional polytope $P \subset \mathbb{R}^n$ is defined as the number $\mathfrak{c}(P) = d - i$, where i is the maximal nonnegative integer satisfying the condition: there exists a sequence $P_0 \subset P_1 \subset \cdots \subset P_i = P$ such that P_j is a pyramid over P_{j-1} for each $1 \le j \le i$. Observe that if P is a rational polytope then so are the polytopes $P_0, P_1, \dots, P_{i-1}$.

Informally, the complexity of a polytope is measured by the number of steps needed to get to the polytope by successively taking pyramids over an initial polytope: the more steps we need the simpler the polytope is. The following are immediately observed:

- the complexity is an invariant of the combinatorial type and it never exceeds the dimension,
- a positive dimensional polytope P is not a pyramid if and only if $\mathfrak{c}(P) = \dim P$,

- simplices are exactly the polytopes of complexity 0,
- we always have the equality $\mathfrak{c}(\mathrm{pyr}(v,P)) = \mathfrak{c}(P)$.

For a cone $C \subset \mathbb{R}^n$ its *complexity* $\mathfrak{c}(C)$ is defined to be $\mathfrak{c}(\Phi(C))$ where $\Phi(C) = \mathcal{G} \cap C$ for any affine hyperplane $\mathcal{G} \subset \mathbb{R}^n$ cross-secting C. For a positive affine monoid $M \subset \mathbb{Z}^n$ its *complexity* $\mathfrak{c}(M)$ is defined to be that of the cone $\mathbb{R}_+ M$.

Consider two polytopes $P \subset Q$, $P \neq Q$. Assume P is obtained from Q by cutting off a pyramid at a vertex $v \in Q$. In other words, $Q = P \cup P'$, $\dim P = \dim P' = \dim Q$ and $P' = \mathrm{pyr}(v, P \cap P')$. In this situation we say that $P \subset Q$ is a *pyramidal extension*. Observe that if $P \subset Q$ is a pyramidal extension then $\dim P = \dim Q \geq 1$.

The following lemma is a key combinatorial fact. Let $P \subset \mathbb{R}^n$ be a polytope. Call a sequence of polytopes $P = P_0, P_1, P_2, \dots$ *admissible* if the following conditions hold for all indices k:

- either $P_{k+1} \subset P_k$ is a pyramidal extension or $P_k \subset P_{k+1}$,
- $P_k \subset P$.

(Observe, $\dim P_k = \dim P_0$ for all k.)

Lemma 5.1. *Let P be a polytope and $U \subset P$ an open subset. There exists an admissible sequence of polytopes $P = P_0, P_1, P_2, \dots$ such that $P_j \subset U$ for all sufficiently large j. If P is rational then the polytopes P_j can be chosen to be rational.*

If P and U are given explicitly (say, by the vertices or support hyperplanes of P and of a simplex inside U). Then there is an algorithm that finds an admissible sequence $P = P_0, P_1, P_2, \dots$

The lemma is proved in [G1] without explicit reference to the algorithmic aspect (see [BrG, §8.G] for the most recent exposition). However, the proof is in fact algorithmic, see for instance [LW].

5.2. **Sufficiency of pyramidal descent.** An extension of monoids $L \subset N$ is called *pyramidal* if:

- $L, N \subset \mathbb{Z}^n$ are nonzero affine positive normal monoids,
- $\Phi(L) \subset \Phi(N)$ is a pyramidal extension of polytopes,
- $N|\Phi(L) = L$.

Here $\Phi(N) = \mathbb{R}_+ N \cap \mathcal{G}$ and for an arbitrarily fixed rational affine hyperplane $\mathcal{G} \subset \mathbb{R}^n$ cross-secting the cone $\mathbb{R}_+ N$.

Observe that if $L \subset N$ is a pyramidal extension then $\mathrm{rank}\, L = \mathrm{rank}\, N \geq 2$.

Let $L \subset N$ be a pyramidal extension of monoids. It will be called *an extension of complexity c* if $\mathfrak{c}\big(\overline{\Phi(N) \setminus \Phi(L)}\big) = c$, where $\overline{\mathcal{Z}}$ refers to the closure of $\mathcal{Z}$ in the Euclidean topology. In this situation we will write $\mathfrak{c}(L \subset N) = c$.

We say that GL_r-*pyramidal descent* holds for a pyramidal extension of monoids $L \subset N$ if for every explicitly given matrix $A \in \mathrm{GL}_r(R[N_*])$ one can effectively find a natural number j and an elementary matrix $E \in \mathrm{E}_r(R[(N_*)^{c^{-j}}])$, together with a representation $E = \prod_k e_{p_k q_k}(\lambda_k)$ where $e_{p_k q_k}(\lambda_k) \in \mathrm{E}_r(R[(N_*)^{c^{-j}}])$, such that $EA \in \mathrm{GL}_r(R[(L_*)^{c^{-j}}])$. We say that GL_r-*pyramidal descent of type c holds for*

monoids of rank m for some $m \in \mathbb{N}$ if GL_r-pyramidal descent holds for all pyramidal extensions of monoids $L \subset N$ with $\mathfrak{c}(L \subset N) = c$ and $\operatorname{rank} N = m$.

We say that $K_{2,r}$-pyramidal descent holds for a pyramidal extension of monoids $L \subset N$ if the homomorphism $K_{2,r}(R[(L_*)^{c^{-\infty}}]) \to K_{2,r}(R[(N_*)^{c^{-\infty}}])$ is surjective. We say that *$K_{2,r}$-pyramidal descent of type c holds for monoids of rank m* for some $m \in \mathbb{N}$ if pyramidal descent holds for all pyramidal extensions $L \subset N$ with $\mathfrak{c}(L \subset N) = c$ and $\operatorname{rank} N = m$.

Proposition 5.2. *Let $M \subset \mathbb{Z}^n$ be an affine positive normal monoid. Then Theorem 1.1(a) (corr. Theorem 1.1(b,c)) holds for the monoid algebra $R[M_*]$ if $K_{2,r}$-pyramidal descent (GL_r-pyramidal descent) of type $< \mathfrak{c}(M)$ holds for monoids of rank $= \operatorname{rank} M$. (Here r is as in the corresponding part of Theorem 1.1.)*

Proof. Let $P_0 \subset P_1 \subset \cdots \subset P_i = \Phi(M)$ be a sequence of rational polytopes where $i = \operatorname{rank}(M) - 1 - \mathfrak{c}(M) = \dim \Phi(M) - \mathfrak{c}(M)$ and $P_j = \operatorname{pyr}(v_j, P_{j-1})$ for each $j \in [1, i]$.

Fix a rational simplex $\Delta \subset P_0$, $\dim \Delta = \dim P_0$. By Lemma 5.1 there is an admissible sequence $P_0 = Q_0, Q_1, Q_2, \dots$ of rational polytopes such that $Q_k \subset \Delta$ for all $k \gg 0$. Then the sequence of polytopes $\tilde{Q}_k = \operatorname{conv}(v_1, \dots, v_i, Q_k)$ is an admissible sequence of rational polytopes such that $\tilde{Q}_0 = \Phi(M)$ and $\tilde{Q}_k$ are contained in the simplex $\tilde{\Delta} = \operatorname{conv}(v_1, \dots, v_i, \Delta)$ for $k \gg 0$. (We assume $\tilde{Q}_k = Q_k$ and $\tilde{\Delta} = \Delta$ when $i = 0$.) Moreover, if $Q_{k+1} \subset Q_k$ is a pyramidal extension then we have

$$\mathfrak{c}\big(\overline{\tilde{Q}_{k+1} \setminus \tilde{Q}_k}\big) \leq \mathfrak{c}(M) - 1.$$

By Gordan's lemma (see Section 3.2) the monoids $\mathbb{R}_+\tilde{Q}_t \cap M$ are all affine. Obviously, they are also normal and positive.

By Corollary 4.2, for Theorem 1.1(a) it is enough to show that

$$K_2(R) = K_{2,r}(R[(M_*)^{c^{-\infty}}]).$$

Let $x \in K_{2,r}(R[(M_*)^{c^{-\infty}}])$. Assume $K_{2,r}$-pyramidal descent of type $< \mathfrak{c}(M)$ holds for monoids of rank $= \operatorname{rank} M$. Then there exist a sequence of elements

$$x_k \in K_{2,r}(R[(M|\tilde{Q}_k)_*]), \quad k = 0, 1, \dots$$

such that:

- $x_0 = x$,
- if $\tilde{Q}_{k+1} \subset \tilde{Q}_k$ is a pyramidal extension for some $k \geq 0$ then x_k is the image of x_{k+1} under the map

$$K_{2,r}(R[((M|\tilde{Q}_{k+1})_*)^{c^{-\infty}}]) \to K_{2,r}(R[((M|\tilde{Q}_k)_*)^{c^{-\infty}}]),$$

- if $\tilde{Q}_k \subset \tilde{Q}_{k+1}$ then x_{k+1} is the image of x_k under the map

$$K_{2,r}(R[((M|\tilde{Q}_k)_*)^{c^{-\infty}}]) \to K_{2,r}(R[((M|\tilde{Q}_{k+1})_*)^{c^{-\infty}}]).$$

In particular, for $k \gg 0$ we have

$$x \in \operatorname{Im}(K_{2,r}(R[((M|\tilde{\Delta})_*)^{c^{-\infty}}]) \to K_{2,r}(R[((M_*)^{c^{-\infty}}]).$$

In view of Theorems 2.1 and 2.7 Proposition 3.2 we are done.

The case of Theorem 1.1(b,c) is treated by the obvious adaptation of the argument above, using the GL_r-pyramidal descent. For the algorithmic issues it is of course important that all the involved convex polyhedral constructions can be carried out effectively. □

In view of the equation (3) in Section 3.3 and Proposition 4.1 we get

Corollary 5.3. *Theorem 1.1 follows if $K_{2,r}$ and GL_r-pyramidal descents hold for any pyramidal extension of monoids, where r is as in the corresponding part of Theorem 1.1.*

Remark 5.4. As mentioned, the concept of a pyramidal descent without consideration of complexities was introduced in [G1]: using induction on $\operatorname{rank} N$, it is shown in [G1] that (unstable) K_0-pyramidal descent holds for all pyramidal extensions $L \subset N$. The complexities were added to the picture in [G5] for reasons not related to this paper at all. However, it is the notion of complexity that makes the induction argument work in Section 7 where we show that, indeed, GL_r and $K_{2,r}$-pyramidal descents hold for all pyramidal extensions $L \subset N$. The argument will use induction on the pairs $(\operatorname{rank} N, \mathfrak{c}(L \subset N))$. In [Mu] this aspect is simply absent.

6. Almost separation

In this section we state the main technical fact to be used in the proof of $K_{2,r}$- and GL_r-pyramidal decents.

Let $M \subset \mathbb{Z}^n$ be an affine positive normal monoid with $\mathrm{gp}(M) = \mathbb{Z}^n$.

Let $\mathcal{H} \subset \mathbb{R}^n$ be a rational hyperplane, dissecting the cone $\mathbb{R}_+M$ into two n-cones $\mathbb{R}_+M = C_1 \cup C_2$. Fix a rational affine hyperplane $\mathcal{G} \subset \mathbb{R}^n$ with $\mathbb{R}_+M = \mathbb{R}_+(\mathbb{R}_+M \cap \mathcal{G})$.

We also fix a real number $\varepsilon > 0$ and a natural number $c \geq 2$.

Let $M_1(\varepsilon) = \mathbb{R}_+M \cap C_1(\varepsilon) \cap M$ and $M_2(\varepsilon) = \mathbb{R}_+M \cap C_2(\varepsilon) \cap M$, where $C_1(\varepsilon)$ and $C_2(\varepsilon)$ refer to the open cones introduced in Section 3.1.

For a ring Λ and a matrix $A \in \mathrm{E}_r(\Lambda)$ under *a representation* $\bar{A}$ we will mean a representation of the form

$$A = \prod_k e_{p_k q_k}(\lambda_k), \quad \lambda_k \in \Lambda.$$

The theorem below is essentially due to Mushkudiani [Mu] (see Remark 6.3):

Theorem 6.1. *Let R be an arbitrary ring, $r \geq 2$ be a natural number and $A \in \mathrm{E}_r(R[M_*])$. Then for any representation $\bar{A}$ one can explicitly find a natural number j_A and a factorization of the form $A = A_1A_2$ for some $A_1 \in \mathrm{E}_r(R[(M_1(\varepsilon)_*)^{c^{-j_A}}])$, together with a representation $\bar{A}_1$, and $A_2 \in \mathrm{SL}_r(R[(M_2(\varepsilon)_*)^{c^{-j_A}}])$.*

(The equality $A = A_1A_2$ is considered in the ambient group $\mathrm{GL}_r(R[M^{c^{-\infty}}])$.)

In other words, the input of the algorithm is an explicit representation of the form $\bar{A} = \prod_k e_{p_k q_k}(\lambda_k)$, $\lambda_k \in R[M_*]$, and the output is a natural number j_A and a factorization $A = A_1A_2$ where $A_1 \in \mathrm{E}_r(R[(M_1(\varepsilon)_*)^{c^{-j_A}}])$ and $A_2 \in \mathrm{SL}_r(R[(M_2(\varepsilon)_*)^{c^{-j_A}}])$,

together with an explicit representation of the form

$$\bar{A}_1 = \prod_k e_{r_k s_k}(\mu_k), \qquad \mu_k \in R[(M_1(\varepsilon)_*)^{c^{-j_A}}].$$

Here it is assumed that in R and M we can explicitly perform the operations.

We want to emphasize that even *without referring the algorithmic aspect*, Theorem 6.1 states a nontrivial fact which leads to the nilpotence of $K_{1,r}(R[M])$.

Remark 6.2. In view of Theorem 1.1(b), Theorem 6.1 is equivalent to the equality

$$\mathrm{E}_r(R[(M_*)^{c^{-\infty}}]) = \mathrm{E}_r(R[(M_1(\varepsilon)_*)^{c^{-\infty}}])\,\mathrm{E}_r(R[(M_2(\varepsilon)_*)^{c^{-\infty}}])$$

which actually explains the name 'almost separation'. However, since Theorem 1.1 is a consequence of Theorem 6.1, we have to resort to the formulation above.

Remark 6.3. Mushkudiani's original version, derived in the course of the proof of [Mu, Theorem 3.1] (but not stated explicitly), claims the existence of a representation of the form $A = A_1A_2$ where $A_1 \in \mathrm{E}(R[(\mathbb{R}_+M \cap C_1 \cap M)^{c^{-\infty}}])$ and $A_2 \in \mathrm{SL}(R[M_2(\varepsilon)^{c^{-\infty}}])$. However, the corrected argument, presented in Section 8, gives the current version. Moreover, the argument in [Mu] never really uses the fact that in Theorem 6.1 one takes iterated cth roots of monomials. But without taking the cth roots of monomials, Theorem 6.1 can *not* hold as it would lead to a contradiction with [G3] and [Sr].

The next theorem is a St-version of Theorem 6.1.

Theorem 6.4. *Let R be a regular ring and $r \geq \max(5, \dim R + 3)$ be a natural number. Then any element $x \in \mathrm{St}_r(R[(M_*)^{c^{-\infty}}])$ has a factorization of the form:*

$$\begin{aligned} x = yz, \qquad & y \in \mathrm{Im}(\mathrm{St}_r(R[(M_1(\varepsilon)_*)^{c^{-\infty}}]) \to \mathrm{St}_r(R[(M_*)^{c^{-\infty}}])), \\ & z \in \mathrm{Im}(\mathrm{St}_r(R[(M_2(\varepsilon)_*)^{c^{-\infty}}]) \to \mathrm{St}_r(R[(M_*)^{c^{-\infty}}])). \end{aligned}$$

The logical scheme of the relationships between Theorems 1.1, 6.1 and 6.4 is given by the following diagram:

$$\begin{tikzcd}
\boxed{\text{Theorem 6.1}} \arrow[r] \arrow[dr] & \boxed{\text{Theorem 1.1(b)}} \arrow[r] \arrow[d] \arrow[dr] & \boxed{\text{Theorem 1.1(c)}} \\
 & \boxed{\text{Theorem 6.4}} \arrow[r] & \boxed{\text{Theorem 1.1(a)}}
\end{tikzcd} \tag{6}$$

which will be realized gradually in the following sections, postponing the proof of Theorem 6.1 to the very end.

Below we explain how Theorems 1.1(b) and Theorem 6.1 together imply Theorem 6.4. This corresponds to the left triangle in diagram (6).

Proof. For simplicity of notation let

$$\begin{aligned} \mathcal{Y} &= \mathrm{Im}(\mathrm{St}_r(R[(M_1(\varepsilon)_*)^{c^{-\infty}}]) \to \mathrm{St}_r(R[(M_*)^{c^{-\infty}}])), \\ \mathcal{Z} &= \mathrm{Im}(\mathrm{St}_r(R[(M_2(\varepsilon)_*)^{c^{-\infty}}]) \to \mathrm{St}_r(R[(M_*)^{c^{-\infty}}])). \end{aligned}$$

First we consider the case when M is simplicial.

Let $E \in \mathrm{E}_r(R[(M_*)^{c^{-\infty}}])$ denote the image of x. By Theorem 6.1 we can write $E = A_1A_2$ where $A_1 \in \mathrm{E}_r(R[(M_1(\varepsilon)_*)^{c^{-\infty}}])$ and $A_2 \in \mathrm{SL}_r(R[(M_2(\varepsilon)_*)^{c^{-\infty}}])$. By Theorem 1.1(b) (or, equivalently, Theorem 3.4(b)) $A_2 \in \mathrm{E}_r(R[(M_2(\varepsilon)_*)^{c^{-\infty}}])\,\mathrm{SL}_r(R)$.

Since R is a retract of the rings $R[(M_*)^{c^{-\infty}}]$, $R[(M_1(\varepsilon)_*)^{c^{-\infty}}]$ and $R[(M_2(\varepsilon)_*)^{c^{-\infty}}]$, we actually have $A_2 \in \mathrm{E}_r(R[(M_2(\varepsilon)_*)^{c^{-\infty}}])$.

By lifting A_1 and A_2 respectively to $\mathcal{Y}$ and $\mathcal{Z}$ we find two elements $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$ such that $x = \xi yz$ for some $\xi \in K_{2,r}(R[(M_*)^{c^{-\infty}}])$. By Proposition 3.2 the monoid $(M_*)^{c^{-\infty}}$ is a filtered union of free monoids. Therefore, Theorems 2.1 and 2.7 imply that $K_{2,r}(R[(M_*)^{c^{-\infty}}]) = K_2(R)$. In particular, $\xi z \in \mathcal{Y}$. Hence the desired representation $x = (\xi y)z$.

Now we consider the case of a general affine positive normal monoid $M \subset \mathbb{Z}^n$.

Fix a surjective monoid homomorphism $\pi : \mathbb{Z}_+^m \to M$ for some m. Its $\mathbb{R}$-linear extension $\mathbb{R}^m \to \mathbb{R}^n$ will be denoted by $\mathbb{R} \otimes \pi$.

There exist a rational hyperplane $\mathcal{H}' \subset \mathbb{R}^m$, dissecting the standard positive orthant $\mathbb{R}_+^m$ into two m-cones $\mathbb{R}_+^m = C_1' \cup C_2'$, and a real number $\varepsilon' > 0$ such that $(\mathbb{R} \otimes \pi)(C_1'(\varepsilon')) \subset C_1(\varepsilon)$ and $(\mathbb{R} \otimes \pi)(C_2'(\varepsilon')) \subset C_2(\varepsilon)$. Here the open convex cones $C_1'(\varepsilon')$ and $C_2'(\varepsilon')$ are considered with respect to arbitrarily fixed affine hyperplane $\mathcal{G}' \subset \mathbb{R}^m$, cross-secting the positive orthant $\mathbb{R}_+^m$. Let $\mathcal{Y}'$ and $\mathcal{Z}'$ denote the sets

$$\mathcal{Y}' = \mathrm{Im}(\mathrm{St}_r(R[(M_1'(\varepsilon')_*)^{c^{-\infty}}]) \to \mathrm{St}_r(R[((\mathbb{Z}_+^m)_*)^{c^{-\infty}}])),$$
$$\mathcal{Z}' = \mathrm{Im}(\mathrm{St}_r(R[(M_2'(\varepsilon')_*)^{c^{-\infty}}]) \to \mathrm{St}_r(R[((\mathbb{Z}_+^m)_*)^{c^{-\infty}}])).$$

Then π induces a surjective group homomorphism

$$\pi_* : \mathrm{St}_r(R[((\mathbb{Z}_+^m)_*)^{c^{-\infty}}] \to \mathrm{St}_r(R[(M_*)^{c^{-\infty}}])$$

such that $\pi_*(\mathcal{Y}') \subset \mathcal{Y}$ and $\pi_*(\mathcal{Z}') \subset \mathcal{Z}$. Therefore, the general case reduces to the case when M is simplicial. $\square$

Remark 6.5. The proof of Theorem 6.4 (in a slightly different formulation) constitutes the main part of [Mu]. It represents a 'Steinberg group version' of the argument in Section 8. However, the approach in [Mu] simply cannot be rescued.

Remark 6.6. As it becomes clear in Section 5, we only need the validity of Theorems 6.1 and 6.4 for the special cuts of $\mathbb{R}_+M$ by $\mathcal{H}$ when one extremal ray of $\mathbb{R}_+M$ lies strictly on one side of $\mathcal{H}$ and the other extremal rays lie on the other side. However, our deduction of Theorem 6.4 from Theorem 6.1 is through lifting the general case to the case when M is simplicial (the map π above) and the mentioned condition on the dissecting hyperplane is in general *not* respected under such a lifting. So we really need the general version of Theorem 6.1.

7. Almost separation implies pyramidal descent

In this section R is a regular ring of finite Krull dimension.

In Section 7.1 we assume $r \geq \max(3, \dim R + 2)$ and show how Theorem 6.1 implies GL_r-*pyramidal descent*. This corresponds to the upper left horizontal arrow

in diagram (6). The upper right arrow simply reflects the fact that the proof of Theorem 1.1(b) is algorithmic in nature.

In Section 7.2 we assume $r \geq \max(5, \dim R + 3)$ and show how Theorems 1.1(b) and 6.4 imply *$K_{2,r}$-pyramidal descent.* This corresponds to the right triangle in diagram (6).

7.1. GL_r**-pyramidal descent.** Here we prove

Lemma 7.1. *GL_r-pyramidal descent holds for any pyramidal extension of monoids.*

Proof. Let $L \subset N$ be a pyramidal extension of monoids in $\mathbb{Z}^n$. We use induction on the pairs $(\operatorname{rank} N, \mathfrak{c}(N))$, ordered lexicographically.

If $\mathfrak{c}(N) = 0$ then N is simplicial and then we are done by Corollaries 4.3 and 4.4. Notice, the condition $\mathfrak{c}(N) = 0$ also includes the case $\operatorname{rank} N \leq 2$.

Now assume $\mathfrak{c}(N) > 0$ and the GL_r-pyramidal descent has been shown for the pyramidal extensions $L' \subset N'$ for which

$$(\operatorname{rank} N', \mathfrak{c}(L' \subset N')) < (\operatorname{rank} N, \mathfrak{c}(L \subset N)).$$

We want to show the equality

$$\mathrm{GL}_r(R[(N_*)^{c-\infty}]) = \mathrm{E}_r(R[(N_*)^{c-\infty}])\, \mathrm{GL}_r(R[(L_*)^{c-\infty}]). \tag{7}$$

By Proposition 5.2 for any affine positive normal monoid M', satisfying the conditions $\operatorname{rank} M' = \operatorname{rank} N$ and $\mathfrak{c}(M') = \mathfrak{c}(L \subset N)$, we have

$$\mathrm{GL}_r(R[(M'_*)^{c-\infty}]) = \mathrm{E}_r(R[(M'_*)^{c-\infty}])\, \mathrm{GL}_r(R). \tag{8}$$

Fix an affine hyperplane $\mathcal{G} \subset \mathbb{R}^n$ cross-secting the cone $\mathbb{R}_+ N$. The Φ-polytopes below are all considered w.r.t. $\mathcal{G}$.

$\Phi(N)$ has exactly one vertex that does not belong to $\Phi(L)$. Call it v. Let $C(v, \Phi(N)) \subset \mathcal{G}$ denote the affine cone spanned by $\Phi(N)$ at v, that is

$$C(v, \Phi(N)) = v + \mathbb{R}_+ \left(\Phi(N) - v\right).$$

We have the rational pyramid $\Delta_1 = \overline{\Phi(N) \setminus \Phi(L)} \subset \Phi(N)$.

Let $\Delta_2 \subset C(v, \Phi(N))$ be any rational pyramid satisfying the conditions:

- $v \in \operatorname{vert}(\Delta_2)$,
- $C(v, \Phi(N)) = v + \mathbb{R}_+ (\Delta_2 - v)$,
- $\Phi(N) \subset \Delta_2$.

The following two conditions are satisfied automatically:

- $\mathfrak{c}(\Delta_2) = \mathfrak{c}(\Delta_1) = \mathfrak{c}(L \subset N)$,
- $\dim \Delta_2 = \dim \Delta_1 = \dim \Phi(N)$.

In particular, (8) implies

$$\mathrm{GL}_r(R[(N_*)^{c-\infty}]) \subset \mathrm{GL}_r(R[((N|\Delta_2)_*)^{c-\infty}]) = \mathrm{E}_r(R[((N|\Delta_2)_*)^{c-\infty}])\, \mathrm{GL}_r(R).\text{[4]} \tag{9}$$

[4]It is here where we need N to be normal – it enables us to consider the monoid $N|\Delta_2$ which satisfies the condition $(N|\Delta_2)|\Phi(N) = N$.

Fix a rational point $\xi \in \operatorname{int}(\Phi(L))$. For a real number λ the homothetic image of a polytope $\Pi \subset \mathcal{G}$ with the factor $\lambda \in \mathbb{R}$ and centered at ξ will be denoted by Π_λ.

For any real number $0 < \lambda < 1$ we fix a real number $\varepsilon_\lambda > 0$ in such a way that

$$\Phi(L)_\lambda(\varepsilon_\lambda) \subset \operatorname{int}(\Phi(L)). \tag{10}$$

Furthermore, for a rational number $0 < \lambda < 1$ we use the notation:

$$N_{1,\lambda}(\varepsilon_\lambda)_* = ((N|(\Delta_1)_\lambda(\varepsilon_\lambda))_*)^{c-\infty} \quad \text{and} \quad N_{2,\lambda}(\varepsilon_\lambda)_* = ((N|(\overline{\Delta_2 \setminus \Delta_1})_\lambda(\varepsilon_\lambda))_*)^{c-\infty},$$

where $(\Delta_1)_\lambda(\varepsilon_\lambda)$ and $(\overline{\Delta_2 \setminus \Delta_1})_\lambda(\varepsilon_\lambda)$ correspondingly refer to the ε_λ-neighborhoods of $(\Delta_1)_\lambda$ and $(\overline{\Delta_2 \setminus \Delta_1})_\lambda$ inside the pyramid $(\Delta_2)_\lambda$.

We record the following consequence of (10):

$$\operatorname{int}(\Phi(N)) \cap (\overline{\Delta_2 \setminus \Delta_1})_\lambda(\varepsilon_\lambda) \subset \operatorname{int}(\Phi(L)). \tag{11}$$

((10) guarantees that the part of $(\overline{\Delta_2 \setminus \Delta_1})_\lambda(\varepsilon_\lambda)$ 'towards v' is in $\operatorname{int}(\Phi(L))$.)

Now by Theorem 6.1 we have

$$\mathrm{E}_r(R[((N|(\Delta_2)_\lambda)_*)^{c-\infty}]) \subset \mathrm{E}_r(R[N_{1,\lambda}(\varepsilon_\lambda)_*])\,\mathrm{SL}_r(R[N_{2,\lambda}(\varepsilon_\lambda)_*])$$

which, in view of (9), implies

$$\mathrm{GL}_r(R[((N|(\Delta_2)_\lambda)_*)^{c-\infty}]) \subset \mathrm{E}_r(R[(N_*)^{c-\infty}])\,\mathrm{GL}_r(R[N_{2,\lambda}(\varepsilon_\lambda)_*]). \tag{12}$$

By letting λ run over the set $\mathbb{Q} \cap (0,1)$, the inclusion (12) implies

$$\mathrm{GL}_r(R[(N_*)^{c-\infty}]) \subset \bigcup_\lambda \mathrm{E}_r(R[(N_*)^{c-\infty}])\,\mathrm{GL}_r(R[N_{2,\lambda}(\varepsilon_\lambda)_*]). \tag{13}$$

Now (7) follows from (13) once we show the following implication for any λ:

$$\begin{cases} A = BC \\ A \in \mathrm{GL}_r(R[(N_*)^{c-\infty}]) \\ B \in \mathrm{E}_r(R[(N_*)^{c-\infty}]) \\ C \in \mathrm{GL}_r(R[N_{2,\lambda}(\varepsilon_\lambda)_*]) \end{cases} \implies \quad C \in \mathrm{GL}_r(R[(L_*)^{c-\infty}]).$$

But for such a triple of matrices, using (11), we have

$$\begin{aligned} C =& B^{-1}A \in \mathrm{GL}_r(R[(N_*)^{c-\infty}]) \cap \mathrm{GL}_r(R[N_{2,\lambda}(\varepsilon)_*]) = \\ & \mathrm{GL}_r(R[(N_*)^{c-\infty} \cap N_{2,\lambda}(\varepsilon)_*]) = \mathrm{GL}_r(R[(N|\operatorname{int}(\Phi(N)) \cap (\overline{\Delta_2 \setminus \Delta_1})_\lambda(\varepsilon_\lambda))^{c-\infty}]) \subset \\ & \mathrm{GL}_r(R[(N|\operatorname{int}(\Phi(L)))^{c-\infty}]) = \mathrm{GL}_r(R[(L_*)^{c-\infty}]). \end{aligned}$$

□

7.2. $K_{2,r}$**-pyramidal descent.** Here we prove

Lemma 7.2. *$K_{2,r}$-pyramidal descent holds for any pyramidal extension of monoids $L \subset N$.*

Proof. We use the same induction as in the proof of Lemma 7.1, that is the induction on the pairs $(\operatorname{rank} N, \mathfrak{c}(N))$, ordered lexicographically. Also, we assume that $L, N \subset \mathbb{Z}^n$.

If $\mathfrak{c}(N) = 0$ then N is simplicial and then we are done by Corollary 4.10. This also includes the case $\operatorname{rank} N \leq 2$.

Now assume $\mathfrak{c}(N) > 0$ and $K_{2,r}$-pyramidal descent has been shown for the pyramidal extensions $L' \subset N'$ for which

$$(\operatorname{rank} N', \mathfrak{c}(L' \subset N')) < (\operatorname{rank} N, \mathfrak{c}(L \subset N)).$$

Pick an arbitrary element $x \in K_{2,r}(R[(N_*)^{c^{-\infty}}])$. We want to show

$$x \in \operatorname{Im}\big(K_{2,r}(R[(L_*)^{c^{-\infty}}]) \to K_{2,r}(R[(N_*)^{c^{-\infty}}])\big). \tag{14}$$

By Proposition 5.2 for any affine positive normal monoid M', satisfying the conditions $\operatorname{rank} M' = \operatorname{rank} N$ and $\mathfrak{c}(M') = \mathfrak{c}(L \subset N)$, we have

$$K_2(R) = K_{2,r}(R[(M'_*)^{c^{-\infty}}]). \tag{15}$$

Fix a rational affine hyperplane $\mathcal{G} \subset \mathbb{R}^n$ cross-secting the cone $\mathbb{R}_+N$. The Φ-polytopes below are all considered w.r.t. $\mathcal{G}$.

We have the pyramid $\Delta = \overline{\Phi(N) \setminus \Phi(L)}$. Fix a rational point $\xi \in \operatorname{int}(\Phi(L))$, a rational number $0 < \lambda < 1$ and a real number $\varepsilon > 0$ so that the following conditions are satisfied[5]:

- x is the image of some $x_\lambda \in K_{2,r}(R[((N_\lambda)_*)^{-\infty}])$ where $N_\lambda = N|\Phi(N)_\lambda$,
- $\Phi(L)_\lambda(\varepsilon) \subset \operatorname{int}(\Phi(L))$
- $\Delta_\lambda(\varepsilon) \subset \Delta'$ for some rational simplex $\Delta' \subset \operatorname{int}(\Phi(N))$, similar to Δ.

Above we have used the notation:

- for any polytope $\Pi \subset \Phi(N)$ its homothetic image with factor λ and centered at ξ is denoted by Π_λ,
- for any polytope $\Pi \subset \Phi(N)$ its ε-neighborhood inside $\Phi(N)$ is denoted by $\Pi(\varepsilon)$.

Consider the monoids $M_1(\varepsilon) = N_\lambda|\Delta_\lambda(\varepsilon)$ and $M_2(\varepsilon) = N_\lambda|\Phi(L)_\lambda(\varepsilon) \subset L_*$. By Theorem 6.4 we have a representation of the form:

$$\begin{aligned}
&x_\lambda = yz,\\
&y \in \operatorname{Im}(\operatorname{St}_r(R[(M_1(\varepsilon)_*)^{c^{-\infty}}]) \to \operatorname{St}_r(R[((N_\lambda)_*)^{c^{-\infty}}])),\\
&z \in \operatorname{Im}(\operatorname{St}_r(R[(M_2(\varepsilon)_*)^{c^{-\infty}}]) \to \operatorname{St}_r(R[((N_\lambda)_*)^{c^{-\infty}}])).
\end{aligned}$$

[5]This can be done first by choosing λ sufficiently close to 1 and then choosing ε sufficiently small, depending on λ.

For the corresponding elementary matrices $E_y, E_z \in \mathrm{E}_r(R[(N_*)^{c^-\infty}])$ we have

$$E_yE_z = 1, \quad E_y \in \mathrm{E}_r(R[(M_1(\varepsilon)_*)^{c^-\infty}]), \quad E_z \in \mathrm{E}_r(R[(M_2(\varepsilon)_*)^{c^-\infty}]),$$

which implies

$$E_y, E_z \in \mathrm{SL}_r(R[(M_1(\varepsilon)_*)^{c^-\infty} \cap (M_2(\varepsilon)_*)^{c^-\infty}]) = \mathrm{SL}_r(R[(M_1(\varepsilon)_* \cap M_2(\varepsilon)_*)^{c^-\infty}])$$

By Theorem 1.1(b) we get

$$E_y, E_z \in \mathrm{E}_r(R[(M_1(\varepsilon)_* \cap M_2(\varepsilon)_*)^{c^-\infty}]).$$

Let

$$w \in \mathrm{Im}\left(\mathrm{St}_r(R[(M_1(\varepsilon)_* \cap M_2(\varepsilon))_*^{c^-\infty}]) \to \mathrm{St}_r(R[((N_\lambda)_*)^{c^-\infty}])\right)$$

be any lifting of E_y. Then we have:

$$\begin{aligned}
&x_\lambda = (yw^{-1}) \cdot (wz),\\
&yw^{-1} \in \mathrm{Im}(\mathrm{St}_r(R[(M_1(\varepsilon)_*)^{c^-\infty}]) \to \mathrm{St}_r(R[((N_\lambda)_*)^{c^-\infty}])),\\
&wz \in \mathrm{Im}(\mathrm{St}_r(R[(M_2(\varepsilon)_*)^{c^-\infty}]) \to \mathrm{St}_r(R[((N_\lambda)_*)^{c^-\infty}])),
\end{aligned}$$

Since the image of yw^{-1} in $\mathrm{E}_r(R[(M_1(\varepsilon)_*)^{c^-\infty}])$ is $\mathbf{1}$ we actually have

$$yw^{-1} \in \mathrm{Im}(K_{2,r}(R[(M_1(\varepsilon)_*)^{c^-\infty}]) \to K_{2,r}(R[((N_\lambda)_*)^{c^-\infty}]))$$

and, similarly,

$$wz \in \mathrm{Im}(K_{2,r}(R[(M_2(\varepsilon)_*)^{c^-\infty}]) \to K_{2,r}(R[((N_\lambda)_*)^{c^-\infty}])).$$

But then the inclusion $M_2(\varepsilon)_* \subset L_*$ implies

$$wz \in \mathrm{Im}(K_{2,r}(R[(L_*)^{c^-\infty}]) \to K_{2,r}(R[(N_\lambda)^{c^-\infty}])).$$

In particular, (14) follows if we show that the image of yw^{-1} in $K_{2,r}(R[(N_*)^{c^-\infty}])$ belongs to $K_2(R)$.

We have

$$\begin{aligned}
&\mathrm{Im}\left(K_{2,r}(R[(M_1(\varepsilon)_*)^{c^-\infty}]) \to K_{2,r}(R[(N_*)^{c^-\infty}])\right) \subset\\
&\mathrm{Im}\left(K_{2,r}(R[((N|\Delta')_*)^{c^-\infty}]) \to K_{2,r}(R[(N_*)^{c^-\infty}])\right)
\end{aligned}$$

and, in view of the conditions $\mathrm{rank}(N|\Delta') = \mathrm{rank}\, N$ and $\mathfrak{c}(N|\Delta') = \mathfrak{c}(L \subset N)$, by (15) we get $K_{2,r}(R[((N|\Delta')_*)^{c^-\infty}]) = K_2(R)$. $\square$

8. Proof of Theorem 6.1

This section presents a corrected version of Mushkudiani's proof of almost separation in $\mathrm{E}_r(R[M])$. The algorithmic part of Theorem 6.1 is a direct consequence of the argument presented below and we do not discuss it separately.

8.1. **Convention and notation.** Here we introduce the notation to be used in the rest of Section 8.

Monoids and cones. We fix an affine positive monoid $M \subset \mathbb{Q}^n$, $n = \operatorname{rank} \operatorname{gp}(M) \geq 2$. We don't require that M is normal or $M \subset \mathbb{Z}^n$. Let $M^+ = M \setminus \{0\}$.

For a point $z \in \mathbb{R}^n$ its nth coordinate will be denoted by z_n.

Assume a rational hyperplane $\mathcal{H} \subset \mathbb{R}^n$ cuts $\mathbb{R}_+M$ into two n-dimensional subcones. Without loss of generality we will assume $\mathcal{H} = \mathbb{R}^{n-1} \oplus 0 \subset \mathbb{R}^n$ – a condition that can be achieved by a rational coordinate change.

We can additionally assume that the cone $\mathbb{R}_+M$ is 'acute' enough to have the following condition satisfied:

$$\forall u, v \in \mathbb{R}_+M \setminus \{0\} \qquad \|u\|, \|v\| < \|u + v\|. \tag{16}$$

In fact, without loss of generality we can assume that no negative multiple of e_1 belongs to M and then (16) can be achieved by applying to M a linear transformation of the form $e_1 \mapsto e_1$ and $e_i \mapsto e_i + ke_1$ with $k \gg 0$ for $i \neq 1$. Here $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

We also fix a rational affine hyperplane $\mathcal{G} \subset \mathbb{R}^n$ such that $\mathbb{R}_+M = \mathbb{R}_+(\mathbb{R}_+M \cap \mathcal{G})$. Thus $\Phi(M) = \mathbb{R}_+M \cap \mathcal{G}$. Recall, for any submonoid $N \subset M$ we put $\Phi(N) = \mathbb{R}_+N \cap \mathcal{G}$.

Monomials. Let R be a ring. *Monomials* in $R[M]$ are simply the elements of M.

The products $a\mu \in R[M]$, $a \in R$, $\mu \in M$ are *terms*. If $a \neq 0$ then μ is called *the support monomial of* $a\mu$. For a nonzero element $\gamma \in R[M]$ the support monomials in the canonical expansion of γ as a sum of terms constitute the set of *the support monomials of* γ. It is denoted by $\operatorname{supp}(\gamma)$.

For a nonzero term $z = a\mu \in R[M]$, $a \in R$, $\mu \in M$, its *length* $\|a\mu\|$ is just the Euclidean norm $\|\mu\|$ in $\mathbb{R}^n$. Let $z_n = \mu_n$.

For a subset $I \subset \mathbb{R}$ we put

$$R[M]_I = \{\gamma \in R[M] \mid \mu_n \in I \quad \text{for every} \quad \mu \in \operatorname{supp}(\gamma)\} \subset R[M].$$

Thus $0 \in R[M]_I$ for any subset $I \subset \mathbb{R}$ and $R \subset R[M]_I$ if $0 \in I$.

For a nonzero term $z = a\mu \in R[M]$, $a \in R$, $\mu \in M$, and a nonzero element $\gamma \in R[M]$ we put $\Phi(z) = \mathcal{G} \cap \mathbb{R}_+\mu$ and $\Phi(\gamma) = \operatorname{conv}\{\Phi(\mu) \mid \mu \in \operatorname{supp}(\gamma)\}$. By convention, $\Phi(0) = \emptyset$. In particular, $\Phi(\gamma)$ is always a polytope inside $\Phi(M)$.

For an element $\gamma \in R[M]$ we say that γ_n (or $\Phi(\gamma)_n$, or $\|\gamma\|$) satisfies certain inequality if the nth coordinate (respectively, the nth coordinate of the Φ-image, the length) of every element $\mu \in \operatorname{supp}(\gamma)$ satisfies the same inequality.

For real numbers $l > 0$ and ε consider the subset

$$\mathcal{B}'(\varepsilon, l) = \{\gamma \in R[M] \mid l \leq \|\gamma\|,\ \varepsilon \leq \Phi(\gamma)_n\} \subset RM^+.$$

Matrices. Fix a natural number $r \geq 2$. For a matrix $A \in \mathcal{M}_r(R[M])$ a *support monomial of* A is by definition a support monomial of some entry of A. The set of support monomials of A is denoted by $\operatorname{supp}(A)$.

For a matrix $A = (\lambda_{ij})_{i,j=1}^r \in \mathcal{M}_r(R[M])$ we say that A_n satisfies certain inequality if every $(\lambda_{ij})_n$ does so.

For real numbers $l > 0$ and ε we introduce the following subsets of $\mathcal{M}_r(R[M])$:

$$\mathcal{A}(\varepsilon) = \{A \in \mathcal{M}_r(R[M]) \mid 1 \notin \operatorname{supp}(A) \text{ and } \varepsilon \leq A_n\},$$
$$\mathcal{B}(\varepsilon, l) = \mathcal{B}'(\varepsilon, l)^{r\times r},$$
$$\mathcal{D} = \{D \in \mathcal{M}_r(R[M]) \mid 1 \notin \operatorname{supp}(D),\ D \text{ is diagonal and } D_n \geq 0\},$$
$$\mathcal{D}_{>0} = \{D \in \mathcal{M}_r(R[M]) \mid D \text{ is diagonal and } D_n > 0\}.$$

Observe that all these matrices have entries from RM^+ and that the zero matrix belongs to each of the mentioned classes of matrices.

As in the previous sections, *a representation* $\bar{E}$ for a matrix $E \in \mathrm{E}_r(R[M])$ means a representation of the form $E = \prod e_{ij}(\gamma_{ij})$, $\gamma_{ij} \in R[M]$. Moreover, we say that $\bar{E}_n$ (resp. $\Phi(\bar{E})_n$) satisfies certain inequality if every $(\gamma_{ij})_n$ (resp. $\Phi(\gamma_{ij})_n$) does so.

8.2. **Commuting rules for elementary matrices.**

Lemma 8.1. *Let ε_1, ε, l be positive real numbers, $i \neq j$ natural numbers, $D \in \mathcal{D}$, and $\alpha, \beta \in R[M]$ nonzero terms. Assume $|\alpha_n| < \varepsilon_1 \leq \beta_n$. Then:*

$$(e_{ji}(\beta) + D)\, e_{ij}(\alpha) = e_{ij}(\alpha) e_{ij}(\gamma)\, (\mathbf{1} + A + B + D')$$

for some

$$\gamma \in R[M]_{[\alpha_n, \varepsilon_1)}, \quad A \in \mathcal{A}(\varepsilon_1), \quad B \in \mathcal{B}(-\varepsilon, l), \quad D' \in \mathcal{D}.$$

Moreover, the support monomials of γ, A, B and D' are products of those of α, β and D.

(In this lemma we don't exclude the case $\alpha \in R$.)

Proof. We want to find $\gamma \in R[M]_{[\alpha_n, \varepsilon_1)}$ and matrices A, B, D' as in the lemma such that

$$e_{ij}(-\gamma) e_{ij}(-\alpha)\, (e_{ji}(\beta) + D)\, e_{ij}(\alpha) = \mathbf{1} + A + B + D'.$$

We have representations of the form:

- $e_{ij}(-\alpha) e_{ji}(\beta) e_{ij}(\alpha) = e_{ij}(a_0) + a_{ji}(\beta) + D_1$ for some $a_0 = -\alpha^2\beta \in R[M]_{(\alpha_n, +\infty)}$ and $\mathcal{D}_1 \in \mathcal{D}$,
- $e_{ij}(-\alpha) D e_{ij}(\alpha) = D + a_{ij}(b_0)$ for some $b_0 \in R[M]_{[\alpha_n, +\infty)}$,
- $a_0 + b_0 = \gamma_1 + a_1 + b_1$ for some $\gamma_1 \in R[M]_{[\alpha_n, \varepsilon_1)} \setminus \mathcal{B}'(-\varepsilon, l)$, $a_1 \in R[M]_{[\varepsilon_1, +\infty)}$, and $b_1 \in \mathcal{B}'(-\varepsilon, l)$.

(Such a representation $a_0 + b_0 = \gamma_1 + a_1 + b_1$ is in general not unique.)

If $\gamma_1 = 0$ then we are done because

$$e_{ij}(-\alpha)\, (e_{ji}(\beta) + D)\, e_{ij}(\alpha) = \mathbf{1} + \big(a_{ij}(a_1) + a_{ji}(\beta)\big) + a_{ij}(b_1) + (D + D_1).$$

So we can assume $\gamma_1 \neq 0$. Then we have representations of the form:

$$e_{ij}(-\gamma_1)\left(e_{ij}(\gamma_1)+D+D_1\right)=e_{ij}(\delta_1)+D+D_1, \quad \delta_1 \in R[M]_{[\alpha_n,+\infty)},$$
$$e_{ij}(-\gamma_1)a_{ji}(\beta)=a_{ji}(\beta)+D_2, \quad D_2 \in \mathcal{D}_{>0}.$$

We can write $\delta_1=\gamma_2+a_2+b_2$ for some

$$\gamma_2 \in R[M]_{[\alpha_n,\varepsilon_1)} \setminus \mathcal{B}'(-\varepsilon,l), \quad a_2 \in R[M]_{[\varepsilon_1,+\infty)}, \quad b_2 \in \mathcal{B}'(-\varepsilon,l).$$

If $\gamma_2=0$ then we are done because

$$e_{ij}(-\gamma_1)e_{ji}(-\alpha)(e_{ji}(\beta)+D)e_{ij}(\alpha)=$$
$$\mathbf{1}+\left[(a_{ij}(a_1+a_2)+a_{ji}(\beta)\right]+a_{ij}(b_1+b_2)+\left[D+D_1+D_2\right].$$

Therefore, there is no loss of generality in assuming that $\gamma_2 \neq 0$. Then we derive elements γ_3, a_3, b_3, δ_2 and a matrix D_3 out from γ_2, a_2, b_2, a_1, b_1 and $D+D_1+D_2$ in the same way a_2, b_2, γ_2, δ_1 and D_2 were derived out from a_1, b_1, γ_1 and $D+D_1$, etc.

If we show that $\gamma_p=0$ for some $p \in \mathbb{N}$ then

$$e_{ij}\left(-\gamma\right)e_{ij}(-\alpha)\left(e_{ji}(\beta)+D\right)=\mathbf{1}+\left[a_{ij}(\alpha')+a_{ji}(\beta)\right]+a_{ij}(\beta')+D'$$

where

$$\gamma=\sum_{k=1}^{p-1}\gamma_k \in R[M|D,\alpha,\beta]_{[\alpha_n,\varepsilon_1)} \setminus B'(-\varepsilon,l),$$
$$\alpha'=\sum_{k=1}^{p}a_k \in R[M]_{[\varepsilon_1,+\infty)}, \quad \beta'=\sum_{k=1}^{p}b_k \in \mathcal{B}'(-\varepsilon,l), \quad D'=\sum_{k=1}^{p}D_k \in \mathcal{D},$$

and the lemma is proved.

Assume to the contrary that $\gamma_p \neq 0$ for all $p \in \mathbb{N}$. On the other hand it follows from the definition of the elements γ_p that every element of $\operatorname{supp}(\gamma_{p+1})$ is *strictly* divisible in M by a some element of $\operatorname{supp}(\gamma_p)$. (In fact, we have $\operatorname{supp}(D), \operatorname{supp}(D_1), \ldots \subset RM^+$ for all $p \geq 1$.) Since M is an affine positive monoid, $\|\gamma_p\| \to \infty$ as $p \to \infty$. But we also have $\gamma_k \in R[M]_{[\alpha_n,+\infty)}$. Therefore, if p is big enough, then the radial direction of the support terms of γ_p are almost parallel to $\mathbb{R}^{n-1} \oplus 0 \subset \mathbb{R}^n$ and, in particular, belong to $\mathcal{B}'(-\varepsilon,l)$.

The claim that the support monomials of γ, A, B and D' are products of those of α, β and D is a consequence of the process of constructing these objects. $\square$

Lemma 8.2. *Let $\varepsilon_1, \varepsilon, l$ be positive real numbers, $A \in \mathcal{A}(0)$ and $B \in \mathcal{B}(-\varepsilon,l)$. Then*

$$E(\mathbf{1}+A+B)=\mathbf{1}+A'+B'+D'$$

for some $A' \in \mathcal{A}(\varepsilon_1)$, $B' \in \mathcal{B}(-\varepsilon,l)$, $D' \in \mathcal{D}$ and $E \in \mathrm{E}_r(R[M])$ with a representation $\bar{E}$ such that $0 \leq \bar{E}_n < \varepsilon_1$. Moreover, the support monomials of A', B', D' and of the factors in $\bar{E}$ are products of the support monomials of A.

Proof. Let $A = (\alpha_{ij})$. For every pair of indices $i \neq j$ we let $\bar{\alpha}_{ij}$ be the sum of those terms in the canonical expansion of α_{ij} that have the nth coordinate $< \varepsilon_1$ and whose length is $< l$. We have a representation of the form

$$\left(\prod_{i\neq j} e_{ij}(-\bar{\alpha}_{ij})\right)(\mathbf{1} + A + B) = \mathbf{1} + A_1 + B_1$$

where:

- the order of factors is chosen arbitrarily,
- $A_1 \in \mathcal{A}(0)$,
- $B_1 \in \mathcal{B}(-\varepsilon, l)$,

The inequality (16) in Section 8.1 implies that $\mathcal{B}(-\varepsilon, l)$ is stable under the multiplication by elementary matrices of the form $e_{ij}(\lambda)$ with $0 \leq \lambda_n$. Therefore, we can repeat the process with respect to the matrix $\mathbf{1} + A_1 + B_1$ etc. The standard elementary matrices that are produced in this process are of the from $e_{ij}(\lambda)$ with $0 \leq \lambda_n < \varepsilon_1$. After p steps we will have a representation of the form

$$E_p(\mathbf{1} + A + B) = \mathbf{1} + A_p + B_p$$

where:

- $A_p \in \mathcal{A}(0)$ and $B_p \in \mathcal{B}(-\varepsilon, l)$,
- $E_p \in \mathrm{E}_r(R[M])$, having a representation $\bar{E}_p$ with $0 \leq (\bar{E}_p)_n < \varepsilon_1$.
- if a support monomial of some non-diagonal entry of A_p has the n-th coordinate $< \varepsilon_1$ and the length $< l$ then it is a product of p elements (maybe with repetitions) of M^+.

Because M is affine positive, the lengths of the products mentioned in the last condition above go to ∞ as $p \to \infty$. In other words, if p is big enough then the mentioned support terms simply do not exist. That is, for p large enough $\mathbf{1} + A_p + B_p = \mathbf{1} + A' + B' + D'$ for some $A' \in \mathcal{A}(\varepsilon)$, $B' \in \mathcal{B}(-\varepsilon, l)$ and $D' \in \mathcal{D}$.

As in the previous lemma, the claim that the support monomials of A', B', D' and of the factors in $\bar{E}$ are products of the support monomials of A and B is a consequence of the process by which these matrices have been constructed. □

To formulate the next result we introduce certain function $\mathfrak{l} : \mathbb{R}^3_{>0} \to \mathbb{R}_{>0}$, where $\mathbb{R}_{>0}$ is the set of positive reals. For a triple $(\varepsilon_1, \varepsilon_2, \varepsilon) \in \mathbb{R}_{>0}$ there exists a real number $l(\varepsilon_1, \varepsilon_2, \varepsilon) > 0$ such that the following implication holds:

$$\begin{aligned} l \geq l(\varepsilon_1, \varepsilon_2, \varepsilon),\ A_1, A_2 \in \mathcal{A}(-\varepsilon_1),\ B \in \mathcal{B}(-\varepsilon_2, l) \implies& \\ A_1B,\ BA_2,\ A_1BA_2 \in \mathcal{B}(-\varepsilon_2 - \varepsilon, l).& \end{aligned} \tag{17}$$

In fact, if $m_1 \in \mathrm{supp}(A_1)$, $m_2 \in \mathrm{supp}(A_2)$ and $x \in \mathrm{supp}(B)$ then the inequality (16) in Section 8.1 implies $|m_1x|, |m_2x|, |m_1m_2x| \geq l$. On the other hand, none of the numbers $\Phi(m_1x)_n$, $\Phi(m_2x)_n$ and $\Phi(m_1m_2x)_n$ can be less than $\Phi(-2\varepsilon_1e_n + x)_n$ (switching do additive notation). Now if $l \gg 0$, depending on ε_1, ε_2 on ε, then $\Phi(-2\varepsilon_1e_n + x)_n$ cannot be less than $\Phi(x)_n - \varepsilon$.

The function $\mathfrak{l}$ is defined by $(\varepsilon_1,\varepsilon_2,\varepsilon)\mapsto l(\varepsilon_1,\varepsilon_2,\varepsilon)$.

Proposition 8.3. *Let:*

- $\varepsilon_1,\varepsilon_2,\varepsilon,l$ *be positive real numbers with* $l\geq\mathfrak{l}(\varepsilon_1,\varepsilon_2,\varepsilon)$,
- $i\neq j$ *be natural numbers,*
- $\alpha\in R[M]$ *be a nonzero term with* $|\alpha_n|<\varepsilon_1$,
- $A\in\mathcal{A}(\varepsilon_1)$, $B\in\mathcal{B}(-\varepsilon_2,l)$ *and* $D\in\mathcal{D}$.

Then:

$$(\mathbf{1}+A+B+D)e_{ij}(\alpha)=e_{ij}(\alpha)E(\mathbf{1}+A_1+B_1+D_1)$$

for some $A_1\in\mathcal{A}(\varepsilon_1)$, $B_1\in\mathcal{B}(-\varepsilon_2-\varepsilon,l)$, $D_1\in\mathcal{D}$ *and* $E\in\mathrm{E}_r(R[M])$, *having a representation* $\bar{E}$ *such that* $\min(\alpha_n,0)\leq\bar{E}_n<\varepsilon_1$. *Moreover, the support monomials of* A_1, B_1, D_1 *and of the factors in* $\bar{E}$ *are products of the support monomials of* α, A, B *and* D.

(Observe, we do not exclude the case $\alpha\in R$.)

Proof. Let β be the ji-entry of A. Then $|\alpha_n|<\varepsilon_1\leq\beta_n$ and by Lemma 8.1 we have a representation of the form

$$(e_{ji}(\beta)+D)\,e_{ij}(\alpha)=e_{ij}(\alpha+\gamma)\,(\mathbf{1}+A'+B'+D') \tag{18}$$

where $\gamma\in R[M]_{[\alpha_n,\varepsilon_1)}$, $A'\in\mathcal{A}(\varepsilon_1)$, $B'\in\mathcal{B}(-\varepsilon_2,l)$ and $D'\in\mathcal{D}$.

We have

$$A''=e_{ij}(\alpha-\gamma)\,(A-a_{ji}(\beta))\,e_{ij}(\alpha)\in\mathcal{A}(0) \tag{19}$$

because

$$\operatorname{supp}(A'')\subset\operatorname{supp}(A)\cup\{\alpha x\mid x\in\operatorname{supp}(A)\}\cup\{\gamma x|\ x\in\operatorname{supp}(A)\}.$$

In view of the implication (17), we also have

$$B''=e_{ij}(-\alpha-\gamma)Be_{ij}(\alpha)\in\mathcal{B}(-\varepsilon_2-\varepsilon,l). \tag{20}$$

Using (18) and the definition of the matrices A'' and B'', we can write:

$$\begin{aligned}&e_{ij}(-\alpha-\gamma)(\mathbf{1}+A+B+D)e_{ij}(\alpha)=\\&e_{ij}(-\alpha-\gamma)\,(e_{ji}(\beta)+D)\,e_{ij}(\alpha)+A''+B''=\\&\mathbf{1}+(A'+A'')+(B'+B'')+D'.\end{aligned}$$

We have $A'+A''+D'\in\mathcal{A}(0)$ by (19) and $B'+B''\in\mathcal{B}(-\varepsilon_2-\varepsilon,l)$ by (20). By Lemma 8.2 we get a representation of the form:

$$E\big(\mathbf{1}+(A'+A''+D')+(B'+B'')\big)=\mathbf{1}+A_1+B_1+D_1$$

where: $A_1\in\mathcal{A}(\varepsilon_1)$, $B_1\in\mathcal{B}(-\varepsilon_2-\varepsilon,l)$, $D_1\in\mathcal{D}$, and $E\in\mathrm{E}_r(R[M])$, having a representation $\bar{E}$ such that $0\leq\bar{E}_n<\varepsilon_1$.

We finally get the desired representation:

$$Ee_{ij}(-\alpha-\gamma)(\mathbf{1}+A+B+D)e_{ij}(\alpha)=\mathbf{1}+A_1+B_1+D_1,$$

that is

$$(\mathbf{1}+A+B+D)e_{ij}(\alpha) = e_{ij}(\alpha)\left(e_{ij}(\gamma)\cdot E^{-1}\right)(\mathbf{1}+A_1+B_1+D_1)\,.$$

That the support monomials of A_1, B_1, D_1 and of the factors in $e_{ij}(\gamma)\cdot E^{-1}$ are products of the support monomials of α, A, B and D follows from the corresponding claims in Lemmas 8.1 and 8.2 and the way these lemmas are used in the argument above. □

8.3. **Almost separation.** Finally, here we prove Theorem 6.1.

In addition to the objects and the conditions on them, listed in Section 8.1, we now require that M is normal and $\mathrm{gp}(M)=\mathbb{Z}^n$.

Also, we extend in the obvious way to the monoid ring $R[(M_*)^{c^{-\infty}}]$ the terminology and notation that was introduced in Section 8.1 for $R[M]$.

Assume $\mathbb{R}_+M = C_1\cup C_2$ where $C_1=\{z\in\mathbb{R}_+M \mid z_n\le 0\}$ and $C_2=\{z\in\mathbb{R}_+M \mid z_n\ge 0\}$.

Fix a real number $\varepsilon>0$. As in Theorem 6.1, we let $M_1(\varepsilon)=\mathbb{R}_+M\cap C_1(\varepsilon)\cap M$ and $M_2(\varepsilon)=\mathbb{R}_+M\cap C_2(\varepsilon)\cap M$.

Let c be a natural number ≥ 2.

We want to prove the inclusion:

$$\mathrm{E}_r(R[(M_*)^{c^{-\infty}}]) \subset \mathrm{E}_r(R[(M_1(\varepsilon)_*)^{c^{-\infty}}])\,\mathrm{SL}_r(R[(M_2(\varepsilon)_*)^{c^{-\infty}}]), \tag{21}$$

the left hand side being considered in $\mathrm{SL}_r(R[(M_*)^{c^{-\infty}}])$.

Lemma 8.4. *For (21) it is enough to consider the matrices* $E=\Pi_{k=1}^{s}e_{i_kj_k}(\alpha_k)$ *where:*

(a) α_k *are terms in* $R[(M_*)^{c^{-\infty}}]$,
(b) $(\alpha_k)_n\in\mathbb{Z}$,
(c) $(\alpha_k)_n<0 \implies (\alpha_k)_n=-1$,
(d) $(\alpha_k)_n>0,\ \beta\in R[(M_*)^{c^{-\infty}}],\ (\alpha_k\beta)_n=1 \implies \alpha_k\beta\in(M_1(\varepsilon)_*)^{c^{-\infty}}$.

Proof. Consider any matrix $E'=\prod_k e_{i_kj_k}(\alpha'_k)\in \mathrm{E}_r(R[(M_*)^{c^{-\infty}}])$. In view of the 1st Steinberg relation (Section 2) we can assume that $\alpha'_k\in R[(M_*)^{c^{-\infty}}]$ are terms. Assume $\alpha'_k=a_k\mu_k$ for some $a_k\in R$ and $\mu_k\in(M_*)^{c^{-\infty}}$. It is enough to consider the matrix $(c^j)_*(E')$ for some $j\gg 0$. Therefore, there is no loss of generality also in assuming that $\mu_k\in M_*$ for all k. Moreover, by taking j sufficiently large we can make the lengths $\|\mu_k\|$ large enough so that the condition (d) is satisfied. In more detail, we have $0\ll\|\alpha'_k\|\le\|\alpha'_k\beta\|$ for any monomial $\beta\in R[(M_*)^{c^{-\infty}}]$, the second inequality being implied by (16) in Section 8.1. But a long monomial with the nth coordinate $=1$ must be almost parallel to the hyperplane $\mathcal{H}=\mathbb{R}^{n-1}\oplus 0$, or equivalently, must belong to the submonoid $(M_1(\varepsilon)_*)^{c^{-\infty}}\subset(M_*)^{c^{-\infty}}$.

At this point we have reached the situation when all but the condition (c) are satisfied. Now the mentioned condition is taken care of as follows.

The normality of M and the equality $\mathrm{gp}(M)=\mathbb{Z}^n$ (equivalently, the condition $M=\mathbb{R}_+M\cap\mathbb{Z}^n$) imply the surjectivity of the monoid homomorphism $M_*\to\mathbb{Z}$,

$\mu \mapsto \mu_n$. Therefore, by Lemma 3.3 for every μ_k with $(\mu_k)_n < 0$ there exists a decomposition of the form (in additive notation):

$$\mu_k = \sum_i \mu_{ki}, \quad \mu_{ki} \in (M_*)^{c-\infty} \cap h^{-1}(-1).$$

Using the 3rd Steinberg relation (Section 2) the matrices $e_{i_k j_k}(\alpha'_k)$ with $(\alpha'_k)_n < 0$ can correspondingly be represented as products of matrices of the form

$$e_{pq}(a_k),\ e_{pq}(\mu_{k1}),\ e_{pq}(\mu_{k2}), \dots$$

Substituting in the product $\prod_k e_{i_k j_k}(\alpha'_k)$ these representations correspondingly for the factors $e_{i_k j_k}(\alpha'_k)$, $(\alpha'_k)_n < 0$, we arrive at the desired representation. $\square$

Proof of the equality (21). Products of elementary matrices of the form mentioned in Lemma 8.4 will be called *admissible representations*.

Let $E \in \mathrm{E}_r(R[(M_*)^{c-\infty}]$, having an admissible representation $\bar{E} = \prod_{k=1}^s e_{i_k j_k}(\alpha_k)$. We want to show

$$E \in \mathrm{E}_r(R[(M_1(\varepsilon)_*)^{c-\infty}])\,\mathrm{SL}_r(R[(M_2(\varepsilon)_*)^{c-\infty}]). \tag{22}$$

Let $M' \subset (M_*)^{c-\infty}$ be the submonoid generated by $\cup_k \mathrm{supp}(\alpha_k)$ and $\tilde{M} \subset (M_*)^{c-\infty}$ be the submonoid generated by $M \cup M'$.

It is important that the elements of $\tilde{M}$ have *integral* nth coordinate.

An admissible representation of E whose factors have support monomials in M' will be called *good*.

Assume $(\alpha_k)_n \le a$ for some $a \ge 0$. Let

$$\alpha_{k_1}, \dots, \alpha_{k_p}, \qquad 1 \le k_1 < k_2 < \cdots < k_p \le s,$$

be determined by the condition:

$$(\alpha_{k_1})_n, \dots, (\alpha_{k_p})_n = a.$$

In this situation we say that the representation $\bar{E}$ is (a,p)*-bounded*.

Consider the lexicographic order on $\mathbb{Z}_+ \times \mathbb{Z}_+$. For any pair (a', p') with $(a,p) \le (a',p')$ we also say that $\bar{E}$ is (a',p')-bounded.

The proof is by induction on the bounding pairs.

If $a = 1$ then (22) follows from the condition (d) in Lemma 8.4: in this situation $E \in \mathrm{E}_r(R[(M_1(\varepsilon)_*)^{c-\infty}])$.

So we can assume $a \ge 2$ and that

$$E \in \mathrm{E}_r(R[(M_1(\varepsilon)_*)^{c-\infty}])\,\mathrm{SL}_r(R[(M_2(\varepsilon)_*)^{c-\infty}])$$

whenever E has an (a',p')-bounded good representation for some $(a',p') < (a,p)$.

It is enough to prove the existence of a representation of the form:

$$\begin{aligned} E = YZ, \quad & Y \in \mathrm{E}_r(R[(M_1(\varepsilon)_*)^{c-\infty}],\ \text{having an } (a',p')\text{-bounded} \\ & \text{good representation } \bar{Y} \text{ for some } (a',p') < (a,p),\ \text{and} \\ & Z \in \mathrm{SL}_r(R[(M_2(\varepsilon)_*)^{c-\infty}]). \end{aligned} \tag{23}$$

There is no loss of generality in assuming that $k_p < s$ for otherwise

$$E = (Ee_{i_s j_s}(-\alpha_s))\, e_{i_s j_s}(\alpha_s)$$

and $Ee_{i_s j_s}(-\alpha_s)$ obviously has an (a', p')-bounded good representation for some $(a', p') < (a, p)$.

Fix positive real numbers ε_2 and ε' so that $\varepsilon_2 + (s - k_p)\varepsilon' = \varepsilon$. Also, fix a real number $l > 0$, sufficiently large with respect to the numbers

$$a,\ \varepsilon_2,\ \varepsilon',\ \varepsilon_2 + \varepsilon',\ \varepsilon_2 + 2\varepsilon',\ \ldots,\ \varepsilon_2 + (s - k_p - 1)\varepsilon'.$$

We apply Proposition 8.3 to the product

$$e_{i_{k_p} j_{k_p}}(\alpha_{k_p}) e_{i_{k_p+1} j_{k_p+1}}(\alpha_{k_p+1})$$

where in the notation of Proposition 8.3:

- the rôle of M is played by $\tilde{M}$,
- $\varepsilon_1 = a$, $\varepsilon_2 = \varepsilon_2$ and $\varepsilon = \varepsilon'$,
- $\mathbf{1} + A + B + D = \mathbf{1} + A + 0 + 0 = e_{i_{k_p} j_{k_p}}(\alpha_{k_p})$,
- $e_{ij}(\alpha) = e_{i_{k_p+1} j_{k_p+1}}(\alpha_{k_p+1})$.

We get

$$e_{i_{k_p} j_{k_p}}(\alpha_{k_p}) e_{i_{k_p+1} j_{k_p+1}}(\alpha_{k_p+1}) = e_{i_{k_p+1} j_{k_p+1}}(\alpha_{k_p+1}) E_1(\mathbf{1} + A_1 + B_1 + D_1)$$

for some $A_1 \in \mathcal{A}(a)$, $B_1 \in \mathcal{B}(-\varepsilon_2 - \varepsilon', l)$, $D_1 \in \mathcal{D}$, and $E_1 \in \mathrm{E}_r(R[\tilde{M}])$, having a good representation $\bar{E}_1$ with $(\bar{E}_1)_n < a$.

Using Proposition 8.3, we can find inductively matrices

$$A_t \in \mathcal{A}(a), \quad B_t \in \mathcal{B}(-\varepsilon_2 - t\varepsilon', l), \quad D_t \in \mathcal{D},$$
$$t \in \{1, \ldots, s - k_p - 1\},$$

starting with the triple A_1, B_1, D_1 above, so that the following holds for each t:

$$(\mathbf{1} + A_t + B_t + D_t) e_{i_{k_p+t} j_{k_p+t}}(\alpha_{k_p+t}) = e_{i_{k_p+t} j_{k_p+t}}(\alpha_{k_p+t}) E_{t+1}(\mathbf{1} + A_{t+1} + B_{t+1} + D_{t+1}),$$

where $A_{t+1} \in \mathcal{A}(a)$, $B_{t+1} \in \mathcal{B}(-\varepsilon_2 - (t+1)\varepsilon', l)$, $D_{t+1} \in \mathcal{D}$, and $E_{t+1} \in \mathrm{E}_r(R[\tilde{M}])$, having a good representation $\bar{E}_{t+1}$ with $(\bar{E}_{t+1})_n < a$.

We have

$$e_{i_{k_p} j_{k_p}}(\alpha_{k_p}) \prod_{t = i_{k_p} + t}^{s} e_{i_t j_t}(\alpha_t) = \mathcal{E}(\mathbf{1} + A_s + B_s + D_s)$$

for some $\mathcal{E} \in \mathrm{E}_r(R[\tilde{M}])$ having a good representation $\bar{\mathcal{E}}$ with $\bar{\mathcal{E}}_n < a$. Hence a representation $E = YZ$ of the form (23) where:

- $Y = \left(\prod_{t=1}^{k_p - 1} e_t(\alpha_t)\right) \mathcal{E}$,
- $Z = \mathbf{1} + A_s + B_s + D_s$. $\square$

References

[BaHS] H. Bass, A. Heller, and R. Swan, *The Whitehead group of a polynomial extension*, Inst. Hautes Études Sci. Publ. Math. **22** (1964), 61–79.

[BrG] W. Bruns and J. Gubeladze, *Polytopes, Rings, and K-theory*, book in preparation. (Preliminary version: http://math.sfsu.edu/gubeladze/publications/kripo.html)

[Coh] P. M. Cohn, *On the structure of the* GL_2 *of a ring*, Inst. Hautes Études Sci. Publ. Math. No. **30** (1966), 5–53.

[Cor] G. Cortiñas, *The obstruction to excision in K-theory and in cyclic homology*, Invent. math. (2005), published electronically.

[G1] J. Gubeladze, *Anderson's conjecture and the maximal monoid class over which projective modules are free*, Math. USSR Sbornik **63** (1989), 165–180.

[G2] J. Gubeladze, *Classical algebraic K-theory of monoid algebras*, Lecture Notes in Math. **1437**, Springer, 1990, 36–94.

[G3] J. Gubeladze, *Nontriviality of $SK_1(R[M])$*, J. Pure Appl. Algebra **104** (1995), 169–190.

[G4] J. Gubeladze, *Higher K-theory of toric varieties*, K-Theory **28** (2003), 285–327.

[G5] J. Gubeladze, *The nilpotence conjecture in K-theory of toric varieties*, Invent. math. **160** (2005), 173–216.

[K1] W. van der Kallen, *Injective stability for K_2,* Lecture Notes in Math. **551**, Springer, 1976, 77–154.

[K2] W. van der Kallen, *Another presentation for Steinberg groups,* Indag. Math. **39** (1977), 304–312.

[LiXW] Z. Lin, L. Xu, and Q. Wu, *Applications of Gröbner bases to signal and image processing: a survey*, Linear Algebra Appl., **391** (2004), 169–202.

[LW] R. Laubenbacher and C. Woodburn, *An algorithm for the Quillen-Suslin theorem for monoid rings*, J. Pure Appl. Algebra **117/118** (1997), 395–429.

[Mu] Z. Mushkudiani, *K_2-groups of monoid algebras over regular rings*, Proc. A. Razmadze Math. Inst., Georg. Ac. Sci. **113** (1995), 120–137.

[Mi] J. Milnor, *Introduction to algebraic K-theory*, Annals of Mathematics Studies **72**, Princeton University Press, 1971.

[PW] H. Park and C. Woodburn, *An algorithmic proof of Suslin's stability theorem for polynomial rings*, J. Algebra **178** (1995), 277–298.

[Q1] D. Quillen, *Higher algebraic K-theory: I*, Lecture Notes in Math. **341**, Springer, 1973, 85–147.

[Q2] D. Quillen, *Projective modules over polynomial rings*, Invent. Math. **36** (1976), 167–171.

[Sr] V. Srinivas, *K_1 of the cone over a curve*, J. Reine Angew. Math. **381** (1987), 37–50.

[Su] A. Suslin, *On the structure of the special linear group over polynomial rings*, Math. USSR Izvestija **11** (1977), 221–238.

[SuW] A. A. Suslin and M. Wodzicki, *Excision in algebraic K-theory*, Ann. Math., II. Ser. **136** (1992), 51–122.

[T] M. Tulenbaev, *The Steinberg group of a polynomial ring*, Mat. Sbornik **117** (1982), 131 – 143. (1982).

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